Stability and error analysis for a numerical scheme to approximate elastic flow

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Abstract

In this thesis we are concerned with the elastic flow of curves in $\mathbb{R}^n$ ($n \geq 2$). The elastic flow is the L^2-gradient flow corresponding to the elastic energy functional, which is the integral of the curvature squared taken along the curve. The elastic energy is not only of mathematical interest, but also has applications in image processing and modeling of biological structures.

The elastic flow leads to a highly nonlinear parabolic system of partial differential equations of fourth order for the position vector of the curve. In order to solve this system numerically, the key idea is to split the fourth order problem into a system of two second order equations for position and curvature vectors. This approach essentially allows one to use piecewise linear finite elements for constructing numerical schemes.

In this thesis we propose and analyze a new fully discrete numerical scheme to approximate the solution of the elastic flow. To our knowledge, there is only an error analysis for a continuous-in-time semidiscrete scheme. We develop our scheme on the basis of the weak form of the elastic flow, using a fully discrete analogue of the variational relation between the position and curvature vectors, differentiated with respect to time. The resulting scheme represents two coupled second order equations and requires solving a nonlinear problem in each time step.

We first use a suitable constrained minimization problem in order to show that the scheme has a unique solution in a proper set. Our main results are error bounds of order $O(h+\Delta t)$ for the position vector in H^1-norm and the curvature vector in L^2-norm and the constants depend on higher norms of the solution to the continuous problem. The error analysis is carried out under a condition that bounds the time step in terms of the spatial grid size. The proof uses an induction argument and is performed in a series of lemmas, in which several energy estimates are derived.

Zusammenfassung

In der vorliegenden Arbeit beschäftigen wir uns mit dem elastischen Fluss von Kurven im $\mathbb{R}^n$ ($n \geq 2$). Der elastische Fluss ist der L^2-Gradientenfluss für die elastische Energie, welche sich als Integral des Quadrates der Krümmung über die Kurve beschreiben lässt. Die elastische Energie ist nicht nur von mathematischem Interesse, sondern hat auch Anwendungen in der Bildverarbeitung und der Modellierung von biologischen Strukturen.

Der elastische Fluss führt zu einem nichtlinearen parabolischen System partieller Differentialgleichungen vierter Ordnung. Die Grundidee, um dieses System numerisch zu lösen, besteht darin, das Problem vierter Ordnung in ein System aus zwei Problemen zweiter Ordnung für den Positions- und Krümmungsvektor zu zerlegen. Diese Methode ermöglicht es, stückweise lineare finite Elemente für die Konstruktion numerischer Schemata zu nutzen.

In dieser Doktorarbeit präsentieren und analysieren wir ein neues volldiskretes numerisches Schema zur Approximation von Lösungen des elastischen Flusses. Bisher gibt es in der Literatur nur Fehlerabschätzungen für ein in der Zeit kontinuierliches semi-diskretes Schema. Wir entwickeln unser Schema auf Basis der schwachen Formulierung für den elastischen Fluss. Dabei wird ein volldiskretes Analogon der nach der Zeit abgeleiteten Variationsgleichung für die Relation zwischen Positions- und Krümmungsvektor verwendet. Das resultierende System besteht aus zwei gekoppelten Gleichungen zweiter Ordnung und benötigt in jedem Zeitschritt die Lösung eines nichtlinearen Problems.

Zunächst nutzen wir ein geeignetes Minimierungsproblem unter Nebenbedingungen, um zu zeigen, dass das Schema eine eindeutige Lösung in einer geeigneten Menge besitzt. Zu den zentralen Ergebnissen der Arbeit gehören Fehlerabschätzungen der Güte $O(h + \Delta t)$ für den Positionsvektor in der H^1-Norm und für den Krümmungsvektor in der L^2-Norm, wobei die Konstanten von höheren Normen der Lösung des kontinuierlichen Problems abhängen. Die Fehleranalyse wird unter einer Bedingung durchgeführt, welche den Zeitschritt abhängig von der Ortsschrittweite beschränkt. Der Beweis verwendet ein Induktionsargument und kombiniert eine Reihe von Lemmata, die mit Hilfe von Energieabschäzungen bewiesen werden.

Contents

Chapter 1

Introduction

1.1 Overview. Problem formulation

Overview

The theory of elasticity started several centuries ago with the tests of Galileo on breaking a beam, held at one end and loaded by a weight at the other. James Bernoulli was the first, who in 1691 precisely formulated a problem of a thin elastic beam [38]. A significant contribution to the study and description of the law of elastic rods was done by Daniel Bernoulli. The scientist found an expression for the bending, also known as elastic, energy stored in a beam. It was noticed that the work, needed to bend an elastic rod, is proportional to the squared curvature [40]

$$E(x) := \frac{1}{2}\int_I \kappa^2 ds,$$

where ds represents the arclength element of a closed curve $x : I \to \mathbb{R}^n$ $(n \geq 2)$ with a curvature κ. It was of particular interest to find critical points of the functional E subject to fixed length, which are called elastic curves, or simply elasticae. Later, D. Bernoulli in his letter to Euler suggested that differential equations, defining an elastica, could be found by minimizing the elastic energy and proposed to express the given problem in variational form [40]. Following these ideas, Euler by means of calculus of variation succeeded in deriving the equations, which mathematically describe an elastica [27]. This leads to the first variation

$$\langle E'(x), \phi\rangle = \int_I \left(\nabla_{ss}^2 y + \frac{1}{2}|y|^2 y\right) \phi \; ds,$$

where ϕ is a periodic on I test function, y is the curvature vector and $\nabla_s f = f_s - (f_s, \tau)\tau$ with $\tau = x_s$ being the unit tangent. At a minimum the functional E necessarily satisfies $E'(x) = 0$, which is equivalent to the equation

$$\nabla_{ss}^2 y + \frac{1}{2}|y|^2 y = 0 \quad \text{in } I.$$

The detailed history of Bernoulli-Euler theory of elastic beams can be found in the works [52] by Todhunter and [40] by Love.

Problem formulation

In this section, we describe the problem to be treated in the thesis and formulate the aims of our research. Furthermore, we review the existing methods currently used to solve this problem and indicate open questions related to the subject.

Let $x : [0, 2\pi] \to \mathbb{R}^n$ $(n \geq 2)$ be a parametrization of a closed curve. Then for $\lambda > 0$ we introduce the modified elastic energy functional

$$E_\lambda(x) := \frac{1}{2} \int_0^{2\pi} \kappa^2 ds + \lambda L(x), \tag{1.1}$$

where $L(x)$ denotes the length of the curve and ds and κ are the arclength element and the curvature, respectively, as already defined above.

One is usually interested in local or global minima of the functional E_λ. A general approach to find critical points is to consider the geometric flow in which the curves evolve according to the negative L^2-gradient of E_λ. The associated evolution equation is given then by

$$x_t = -\bigtriangledown^2_{ss} y - \frac{1}{2}|y|^2 y + \lambda y \tag{1.2}$$

and represents a highly nonlinear fourth-order parabolic system of partial differential equations (PDEs).

While linear PDEs are well understood and exact solutions established in many cases, analytical treatment of nonlinear PDEs is more difficult task and rarely leads to the exact solution. Therefore one seeks to solve this type of equations numerically and estimate the error of the corresponding approximate solution. Numerical methods for solving (1.2) frequently use a decoupling strategy. The idea of decoupling is to split the problem into two second order problems for the position and curvature vectors, where only first derivatives of the entering functions including test functions appear. This allows one to use piecewise linear finite elements.

The main goal of the thesis is to derive a novel fully discrete, i.e. in space and time, numerical scheme to approximate the solution of (1.2) and carry out the error analysis for the resulting scheme. To our knowledge, there is no such result in the literature. Furthermore, in our work we prove the existence of the unique discrete solution under certain conditions on the time step step size with respect to the spatial grid size. The error bounds for the introduced scheme are satisfied under these conditions as well. We note that this sort of restriction is a common case for the time-dependent problems.

Variational formulation. Numerical scheme

Variational formulation

As already mentioned in the first section, Euler formulated the elastica problem using a variational principle. Our fully discrete numerical scheme is based on the discretization of the continuous one and takes advantage of variational techniques as well. Therefore, we outline here the derivation of the variational formulation for the continuous problem, leading to our numerical scheme. To this end, we adapt the approach from [25] to the case of curves.
Let us consider a curve $x = x(u)$ not necessarily parametrized by arclength. We define the tangent and curvature vectors in a standard way

$$\tau = \frac{x_u}{|x_u|}, \quad y = \frac{1}{|x_u|}\left(\frac{x_u}{|x_u|}\right)_u = \frac{\tau_u}{|x_u|}, \tag{1.3}$$

so that the energy functional (1.1) becomes

$$E_\lambda(x) = \frac{1}{2}\int_0^{2\pi} |y|^2|x_u| + \lambda \int_0^{2\pi} |x_u|. \tag{1.4}$$

Then the first variation of E_λ in direction $\phi \in H^1_{per}\left((0,2\pi),\mathbb{R}^n\right)$ is given by

$$\langle E'_\lambda(x), \phi\rangle := \frac{d}{d\varepsilon}E_\lambda(x+\varepsilon\phi)_{|\varepsilon=0} = \int_0^{2\pi} (y, y_\phi)\,|x_u| + \frac{1}{2}\int_0^{2\pi} |y|^2(\tau,\phi_u) + \lambda\int_0^{2\pi}(\tau,\phi_u),$$

where by y_ϕ we have denoted the derivative of y in direction ϕ and $H^1_{per}\left((0,2\pi),\mathbb{R}^n\right)$ represents the space of all periodic in $H^1\left((0,2\pi),\mathbb{R}^n\right)$ functions. In order to find y_ϕ we use the relation between position and curvature vectors in variational form

$$\int_0^{2\pi} (y,\psi)\,|x_u| + \int_0^{2\pi} (\tau,\psi_u) = 0 \quad \forall\psi \in H^1_{per}((0,2\pi),\mathbb{R}^n)$$

and its first variation in direction $\phi \in H^1_{per}\left((0,2\pi),\mathbb{R}^n\right)$

$$\int_0^{2\pi} (y_\phi,\psi)\,|x_u| + \int_0^{2\pi} (y,\psi)(\tau,\phi_u) + \int_0^{2\pi} \frac{1}{|x_u|}(P\phi_u,\psi_u) = 0, \tag{1.5}$$

where P is the projection matrix defined as

$$P = I_n - \tau\otimes\tau. \tag{1.6}$$

Inserting $\psi = y$ as a test function into (1.5) yields

$$\langle E'_\lambda(x),\phi\rangle = -\int_0^{2\pi}\frac{1}{|x_u|}(Py_u,\phi_u) - \frac{1}{2}\int_0^{2\pi}|y|^2(\tau,\phi_u) + \lambda\int_0^{2\pi}(\tau,\phi_u).$$

Then, for the time-depending functions $x, y : [0, 2\pi] \times [0, T] \to \mathbb{R}^n$ the weak form of the gradient flow corresponding to the functional E_λ reads as

$$\int_0^{2\pi} (x_t, \phi)\, |x_u| - \int_0^{2\pi} \frac{1}{|x_u|} (P y_u, \phi_u) - \frac{1}{2} \int_0^{2\pi} |y|^2 (\tau, \phi_u) + \lambda \int_0^{2\pi} (\tau, \phi_u) = 0, \tag{1.7}$$

$$\int_0^{2\pi} (y, \psi)\, |x_u| + \int_0^{2\pi} (\tau, \psi_u) = 0, \tag{1.8}$$

for all $\phi, \psi \in H^1_{per}((0, 2\pi), \mathbb{R}^n)$ and $0 \le t \le T$. For a smooth function x one can verify that (1.7)-(1.8) are equivalent to (1.2).
Let us now show that $t \mapsto E_\lambda\,(x(\cdot, t))$ is decreasing. Differentiating (1.8) with respect to time, we receive

$$\int_0^{2\pi} (y_t, \psi)\, |x_u| + \int_0^{2\pi} (y, \psi)(\tau, x_{tu}) + \int_0^{2\pi} \frac{1}{|x_u|} (P x_{tu}, \psi_u) = 0.$$

Setting now $\psi = y$ in the above equation and $\phi = x_t$ in (1.7), we immediately obtain the energy decrease

$$\begin{aligned}
\partial_t E_\lambda(x) =& \int_0^{2\pi} (y_t, y)\, |x_u| + \frac{1}{2} \int_0^{2\pi} |y|^2 \,(\tau, x_{tu}) + \lambda \int_0^{2\pi} (\tau, x_{tu}) \\
=& \frac{d}{dt} \left(\frac{1}{2} \int_0^{2\pi} |y|^2\, |x_u| + \lambda \int_0^{2\pi} |x_u| \right) \\
=& - \int_0^{2\pi} |x_t|^2\, |x_u| \le 0.
\end{aligned}$$

Numerical scheme

In the current section, we present a novel scheme, which lies in the heart of our research and is analyzed in the later chapters.
Our starting point was a result obtained by Deckelnick and Dziuk in [22]. The authors analyzed a semidiscrete numerical scheme, in the sense that it is discrete in space and continuous in time, for approximating the evolution of parametric curves by elastic flow in $\mathbb{R}^n$. The authors proved the following error bounds

$$\sup_{t \in [0,T]} \|x(\cdot, t) - x_h(\cdot, t)\|^2_{H^1} + \int_0^T \|x_t(\cdot, t) - x_{ht}(\cdot, t)\|^2\, dt \le C h^2,$$

$$\sup_{t \in [0,T]} \|y(\cdot, t) - y_h(\cdot, t)\|^2_{H^1} + \int_0^T \|y_u(\cdot, t) - y_{hu}(\cdot, t)\|^2\, dt \le C h^2,$$

where the constant C depends only on $T > 0$ and certain norms of the continuous solution x of (1.2), whereas x_h and y_h represent the solutions of the semidiscrete problem

$$\begin{aligned}
\int_0^{2\pi} I_h \left[(x_{ht}, \phi_h)\right] |x_{hu}| - \int_0^{2\pi} \frac{1}{|x_{hu}|} \left(P_h y_{hu}, \phi_{hu}\right) - \frac{1}{2} \int_0^{2\pi} I_h \left[|y_h|^2\right] (\tau_h, \phi_{hu}) \\
+ \lambda \int_0^{2\pi} (\tau_h, \phi_{hu}) = 0,
\end{aligned} \tag{1.9}$$

$$\int_0^{2\pi} I_h\left[(y_h,\psi_h)\right]|x_{hu}| + \int_0^{2\pi} (\tau_h,\psi_{hu}) = 0, \quad (1.10)$$

corresponding to (1.7)-(1.8). Here, ϕ_h, ψ_h are test functions of a suitable finite element space with a grid size h. By I_h we have denoted the Lagrange interpolation operator, while $P_h = I_n - \tau_h \otimes \tau_h$ and $\tau_h = \dfrac{x_{hu}}{|x_{hu}|}$ are the projection matrix and the tangent vector, respectively. In order to perform test calculations, the authors used a semi-implicit time discretization. However, there is no analysis for the resulting fully discrete scheme.
In this context, we suggest the following fully discrete numerical scheme to approximate the elastic flow of a curve

$$\begin{aligned}\int_0^{2\pi} I_h\left[\left(\frac{x_h^{m+1}-x_h^m}{\Delta t},\phi_h\right)\right]|x_{hu}^m| - \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1},\phi_{hu}\right)}{\left|x_{hu}^{m+1}\right|} \\ -\frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^{m+1}\right|^2\right]\left(\tau_h^{m+1},\phi_{hu}\right) + \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1}-x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\phi_{hu}\right) \quad (1.11) \\ +\lambda\int_0^{2\pi}\left(\tau_h^{m+1},\phi_{hu}\right) = 0,\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} I_h\left[\left(y_h^{m+1},\psi_h\right)\right]\left|x_{hu}^{m+1}\right| - \int_0^{2\pi} I_h\left[(y_h^m,\psi_h)\right]|x_{hu}^m| \\ +\int_0^{2\pi}\frac{\left(P_h^m\left(x_{hu}^{m+1}-x_{hu}^m\right),\psi_{hu}\right)}{\left|x_{hu}^{m+1}\right|} = 0. \quad (1.12)\end{aligned}$$

Here for a generic function z we used $z^m = z(\cdot, m\Delta t)$, $m = 0, ..., M$ with a final time $T = M\Delta t > 0$. Choice of the initial data is discussed in Section 2.2. For a detailed explanation and definition of the notations, used above, we refer the reader to Section 2.2, where we introduce discrete quantities.
The first equation of our scheme represents a time-discrete version of (1.9) with the exception of

$$\int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1}-x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\phi_{hu}\right).$$

This term vanishes as $\Delta t \to 0$ and its role will become clear in the derivation of the existence and uniqueness result in Section 3.1. The second equation (1.12) of the scheme is a discrete analogue of the equation (1.10) differentiated with respect to time

$$\begin{aligned}\frac{d}{dt}\left(\int_0^{2\pi} I_h\left[(y_h,\psi_h)\right]|x_{hu}| + \int_0^{2\pi}(\tau_h,\psi_{hu})\right) \\ = \int_0^{2\pi}\partial_t\left(I_h\left[(y_h,\psi_h)\right]|x_{hu}|\right) + \int_0^{2\pi}\frac{(P_h x_{htu},\psi_{hu})}{|x_{hu}|}.\end{aligned}$$

Remember that this relation was important in deriving the energy decrease for continuous problem and also holds for the semidiscrete case. We note that the chosen scheme represents two coupled second order equations and requires solving a nonlinear problem in each time step.

In this connection, we formulate the main result of our work.

Theorem 1.1. *Let $x : [0, 2\pi] \times [0, T] \to \mathbb{R}^n$ be a sufficiently smooth solution of (1.2). Then there exists a solution (x_h^m, y_h^m), $m = 0, ..., M$ of (1.11)-(1.12) and the following error bounds are satisfied*

$$\max_{m=0,...,M} \left(\|x^m - x_h^m\|_{H^1}^2 + \|y^m - y_h^m\|^2 \right) + \sum_{m=0}^{M-1} \Delta t \left(\left\| y_u^{m+1} - y_{hu}^{m+1} \right\|^2 + \left\| \frac{e^{m+1} - e^m}{\Delta t} \right\|^2 \right) \leq C \left(h^2 + \Delta t^2 \right), \tag{1.13}$$

where $e^m = x^m - x_h^m$. The constant C depends only on the norms of the solution of the continuous problem and does not depend on the spatial grid size h and time step size Δt. It should be also remarked that error bounds (1.13) are satisfied under certain conditions on h and Δt, which will be formulated in the later chapters.

1.2 Research studies. Willmore functional

This section gives a brief overview of existing methods for solving elastic energy problems. Additionally, we mention here the strategies listed in the literature for the Willmore functional, which can be thought as a higher dimensional case of the posed problem. Moreover, we give some examples on the applications of the elastic rod theory and curvature-based energy functionals.

Research studies

Curve evolutions under the gradient flow for elastic energy have been intensively investigated in the literature. Also different approaches have been introduced to solve the associated evolutionary equation.
Polden in [45] studied the evolution of closed curves in the plane under the flow (1.2). The author proved the long-time existence of smooth solutions for (1.2) with λ being chosen positive to penalize the length growth. This result has been extended by Dziuk, Kuwert and Schätzle in [26] for curves in arbitrary dimension. In this study, the length of the curve is either penalized as well or fixed as a constraint. The authors also proved the subconvergence of solutions to an elastica, after the curve has been reparametrized by its arclength and translated. Moreover, Dziuk et al. derived an algorithm for the numerical approximation of the solution. To this purpose, the authors split the fourth-order problem into a system of two second-order equations for position and curvature vectors. The second order convergence in the maximum norm for both vectors was revealed in the numerical test calculations. An error analysis of the elastic flow of parametric curves in $\mathbb{R}^n$ can be found in [22]. In [2, 5] Barrett, Garcke and Nürnberg presented an alternative scheme,which particularly ensures an equidistribution of points along the curve.

Since we are interested in closed curves, we only shortly mention some results on the boundary value problems, which are few up to now in the literature. The motion of open curves with clamped boundary conditions in $\mathbb{R}^n$ under the elastic flow has been studied by Lin (see [39]). Dall'Acqua and Pozzi in [20] extended the Helfrich energy for closed curves to n-dimensional case with $n \geq 2$ and studied motion of open regular curves with fixed boundary points under the L^2-gradient flow for the corresponding functional. Recent work [19] by three abovementioned authors extends a long-time existence result presented in [26] to the case of open curves with natural boundary conditions. Furthermore, in [19] a subconvergence proof is also provided.
All above mentioned numerical methods discretize the problem semi-implicit in time and thus require to solve at each time step a linear system of equations. As a consequence, one has to impose a grid-dependent condition on the time step size of the kind $\Delta t \leq Ch^2$. Motivated by this shortcoming, Balzani and Rumpf in [1] and Perl, Pozzi and Rumpf in [44] used a different approach, which was earlier introduced by Olischläger and Rumpf in [43]. They considered a nested variational time discretization with an inner minimization problem to be solved in each time step. An advantage of such approach is that one immediately obtains the decrease of the discrete energy. Computational results demonstrated the robustness of the proposed numerical scheme, which essentially allows the time step size of order of the spatial grid size.

Willmore functional

The Willmore functional, defined as the integral over the surface of the mean curvature squared

$$W(\Gamma) = \int_\Gamma H^2 dA,$$

originates in the Euler-Bernoulli theory of elastic rods and represents the elastic energy for surfaces. It was first introduced by Germain in 1810, but a broad attention was attracted much later in Willmore's work [54]. Critical points of the Willmore functional are referred to as Willmore surfaces. Theoretical results on the Willmore energy, including existence of closed Willmore surfaces of prescribed genus as well as local and global existence for the Willmore flow of closed surfaces, can be found in [49, 6] and [33, 50, 35, 34], respectively, while regularity result has been obtained in [46].
Experiments on the surface motion by the L^2-gradient flow have been carried out in [31] using Brakke's surface Evolver (see [13]), providing the evidence that singularities can be developed in finite time. A finite difference scheme to approximate axisymmetric solutions is presented in [41].
Let us now mention some parametric finite element methods. Authors in [4] numerically approximate with piecewise linear continuous finite elements the Willmore flow, Gauß curvature flow and some generalizations of them, such as Helfrich flow. The presented numerical algorithms extend their previous results (see [3]) and differ from other existing schemes by a good distribution of mesh points. Another approximation of the elastic flow of closed two-dimensional surfaces in $\mathbb{R}^3$ by linear elements is done by Dziuk

[25] and Rusu [47], where the latter work is analogous to the results in [26] for curves. Approximation by quadratic elements is performed in [12] and shows its robustness with respect to the mesh distribution (see [11]). All these methods use the splitting strategy, i.e decoupling the fourth-order problem into two second order equations.
While abovementioned papers treat closed surfaces, existence of the Willmore surfaces with prescribed boundaries has been less investigated. We refer the reader to [42] by Nitsche, where the appropriate boundary conditions are discussed and some existence results are provided as well. Authors in [15] generalized the result of Rusu in [47] for surfaces with boundaries. Willmore surfaces of revolution satisfying the Dirichlet boundary conditions have been investigated in [17, 18]. A level set method to Willmore flow is derived in [23].

Applications

Theory of elastic rods is applicable to different fields of science. In biology, due to the definition of an elastic rod as a structure with length much greater than diameter, it is used in modeling polymers, bacterial fibers and DNA conformations [7, 8, 9, 53, 32]. Thin long structures also play an important role in physics [51, 24]. In mathematics, connections between space curves and partial differential equations are of interest [36, 37].

The Willmore functional is applied in surface restoration, where a destroyed region has to be replaced in a suitable way by a surface patch, see [10, 16, 43]. The Helfrich functional is used to describe the behavior of red blood cell membranes. It was first mentioned in 1970 by Canham [14] and later in 1973 was specified by Helfrich [29]. Cell membrane consists of the lipid bilayer, which has a thickness that is much smaller than the size of the cell. Therefore membranes can be considered as two-dimensional surfaces, embedded in three-dimensional space [48]. Being deformed, cell membranes tend to the equilibrium state, trying to minimize the free elastic energy. This energy was proposed by Helfrich in [29], based on the similarities between lipid bilayers and liquid crystals. Configurations of cell membranes arise due to the bending elasticity, as introduced independently in three papers [14, 29, 28], and can be expressed in terms of the curvature of the membrane surface.

1.3 Thesis outline

The aim of our research is to carry out an error analysis for (1.11)-(1.12), which constitutes the major part of our work. The proof of the error bounds is done along the lines of [22], where a semidiscrete scheme for the elastic flow is analyzed, using 7 auxiliary lemmas. We have extended these lemmas for the fully discrete case and proved the error bounds (1.13).

In Chapter 2 we introduce a finite element space and define some related notations. Additionally, we present here important theorems and inequalities, interpolation and inverse estimates, which will be used in our further analysis.

In Chapter 3 we are concerned with the unique solvability of the equations (1.11)-(1.12) of our numerical scheme. We aim to prove the existence of the unique solution to the posed problem in a certain admissible set. To accomplish this goal, which we formulate in Theorem 3.1, we use a suitable constrained minimization problem. First, we prove the existence of the minimizer in a certain class of admissible functions and then show that this minimizer solves our problem, i.e. satisfies the equations (1.11)-(1.12). Afterwards, we prove that the system of equations (1.11)-(1.12) is uniquely solvable, for what we take the difference between two solutions and show that this difference is equal to zero. Further, in Lemma 3.7 we consider the second equation of our scheme and derive a different representation for it. Due to the fully discreteness of the scheme this gives rise to the remainder term R_h^{m+1} (the exact form is given in the abovementioned lemma). It is worth noting that estimation of the remainder term requires particular treatment (is done in Chapter 5) and its presence complicates the error analysis.

Chapter 4 is devoted to the main result of the thesis – the error analysis of the numerical scheme, which has been introduced above. In order to control expressions on the left-hand side of (1.13), we insert into the equations (1.11)-(1.12) of our scheme suitable test functions and derive in Lemma 4.4 and Lemma 4.5 the estimates for the norms $\left\|x^{m+1} - x_h^{m+1}\right\|$, $\left\|y^{m+1} - y_h^{m+1}\right\|$, respectively. The combined result is presented in Lemma 4.6. Some quantities, which appear on the right-hand side of the above lemma, we cannot handle directly using the Gronwall argument in Lemma 4.12. Therefore additional estimates are required. The main difficulty represents the estimation of the difference of the length elements. We note that from (1.2) follows $(x_t, \tau) = 0$, i.e. the equation degenerates in tangential direction. Thus, in order to estimate the first derivative of the position vector x, one has to gain control on the direction (see Lemma 4.7) and the length separately, what was done in Lemma 4.11 and Lemma 4.12. The first derivative of the curvature vector is treated in Lemma 4.10. In order to close the error analysis, we formulate in Theorem 4.2 the induction-type argument.

In Chapter 5 we complete the proof of Theorem 4.2 and justify the assumptions on the discrete solution, made in Theorem 3.1. In this chapter we also estimate the remainder term, first introduced in Lemma 3.7.

We summarize our work and give some remarks related to the scheme and possible modifications of the analysis in the last Chapter 6.

Derivation of auxiliary results is collected in the Appendices for the sake of readability. Appendix A includes 4 additional lemmas, which give a detailed proof for the results used in Lemma 4.11. In the second Appendix B we carry out the estimates of certain remainder terms, required for the proof of Lemma 4.12 and appeared due to the discretization of the length element in Lemma 4.11.

Chapter 2

Analytic Foundations. Discretization. Initial conditions

2.1 Analytic foundations

In this section, we fix some notations and collect well-known theorems and important inequalities, which will be used in the error analysis.
Let us denote by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^1}$ the norm of $L^p(0,2\pi)$ and $H^1(0,2\pi)$, respectively

$$\|u\|_{L^p} = \left(\int_0^{2\pi} |u(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$
$$\|u\|_{L^\infty} = \operatorname*{ess\ sup}_{x\in(0,2\pi)} |u(x)|, \quad p = \infty,$$
$$\|u\|_{H^1} = \left(\int_0^{2\pi} |u(x)|^2 dx + \int_0^{2\pi} |u'(x)|^2 dx\right)^{\frac{1}{2}}.$$

For $p = 2$ we write $\|\cdot\|_{L^2} = \|\cdot\|$.
In the further analysis we use the following well-known inequalities and theorems.
Cauchy-Schwarz inequality. Let $f, g \in L^2(0,2\pi)$. Then the following inequality holds:

$$\int_0^{2\pi} |f(x)||g(x)|dx \le \|f\|\|g\|. \tag{2.1}$$

Cauchy's inequality for sums. Let $x_i, y_i \in \mathbb{R}$, $i = 1,...,n$ then

$$\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}}. \tag{2.2}$$

Young's inequality. For $a, b \in \mathbb{R}_{\ge 0}$ and $\varepsilon > 0$ holds:

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2. \tag{2.3}$$

Theorem 2.1 (Gronwall's inequality, [30])**.** *Let $y(t), f(t)$, and $g(t)$ be non-negative continuous functions on $[0,T]$ and let us assume that for $0 \leq t \leq T$ we have*

$$y(t) \leq f(t) + \int_0^t g(s)y(s)ds.$$

Then for $0 \leq t \leq T$ we also have

$$y(t) \leq f(t) + \int_0^t g(s)f(s)\ exp\left(\int_s^t g(u)du\right) ds.$$

Theorem 2.2 (Discrete Gronwall's Lemma, [30])**.** *If $\langle y_n \rangle, \langle f_n \rangle$, and $\langle g_n \rangle$ are non-negative sequences and*

$$y_n \leq f_n + \sum_{0 \leq k < n} g_k y_k \quad \text{for } n \geq 0,$$

then

$$y_n \leq f_n + \sum_{0 \leq k < n} g_k f_k\ exp\left(\sum_{k<j<n} g_j\right) \text{ for } n \geq 0. \tag{2.4}$$

2.2 Discretization

We define in this section a finite element space and introduce related notations. Furthermore, we present some useful relations, which hold in the chosen discrete space.

Space discretization

Let $0 = u_0 < u_1 < ... < u_{N-1} < u_N = 2\pi$ be a partition of $[0, 2\pi]$ into subintervals $I_j = [u_{j-1}, u_j]$ of length $h_j := u_j - u_{j-1}$ and set $h := \max_{j=1,...,N} h_j$. The following inverse assumption is required

$$h \leq \hat{c} h_j \quad \text{for all } j = 1, ..., N, \tag{2.5}$$

where $\hat{c}$ is independent of h. Let us denote by X_h the space of linear finite elements

$$X_h := \left\{ \eta_h \in C^0\left([0, 2\pi]\right) | \eta_{h|I_j} \text{ is a linear polynomial}, j = 1, ..., N \text{ and } \eta_h(0) = \eta_h(2\pi) \right\},$$

generated by the nodal basis functions φ_j, $j = 1, ..., N$, defined by

$$\varphi_j(u) = \begin{cases} \dfrac{u - u_{j-1}}{h_j} & \text{if } u \in [u_{j-1}, u_j], \\ \dfrac{u_{j+1} - u}{h_{j+1}} & \text{if } u \in [u_j, u_{j+1}], \\ 0 & \text{otherwise,} \end{cases} \qquad \varphi_j(u_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \quad i, j = 1, ..., N. \end{cases}$$

The basis function φ_j, $j = 1, ..., N$ is shown in Fig. 2.1.

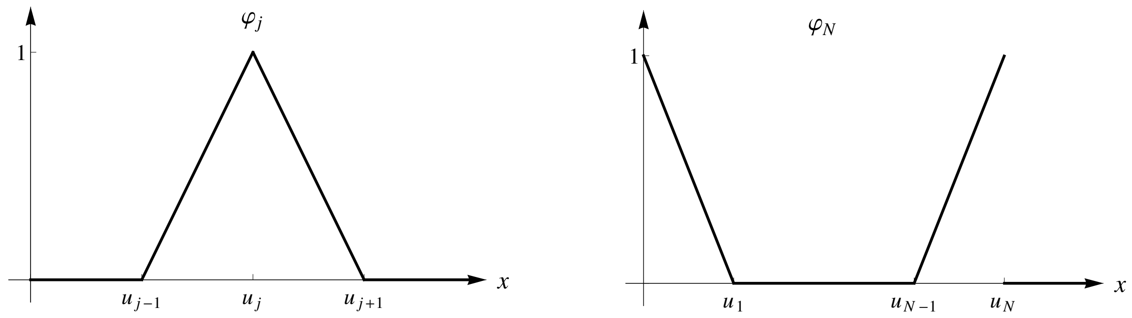

Figure 2.1: On the left the basis function φ_j, $j = 1, ..., N-1$, on the right φ_N.

Next, in order to deal with vector-valued functions we introduce the following space

$$X_h^n := \{\phi_h : [0, 2\pi] \to \mathbb{R}^n | \phi_{h,i} \in X_h, i = 1, ..., n\}.$$

We denote by I_h the Lagrange interpolation operator

$$I_h f = \sum_{i=1}^{N} f(u_i)\varphi_i.$$

Important inequalities

The following interpolation estimates hold

$$\|f - I_h f\| + h\|f_u - (I_h f)_u\| \le Ch^2\|f\|_{H^2}, \quad \forall f \in H^2_{per}(0, 2\pi), \tag{2.6}$$

where $H^2_{per}(0, 2\pi)$ denotes the space of all periodic functions in $H^2(0, 2\pi)$.
We present further useful relations:

$$\int_{I_j} |\phi_h|^2 \le \int_{I_j} I_h[|\phi_h|^2] \le C\int_{I_j} |\phi_h|^2, \tag{2.7}$$

$$\int_{I_j} (\phi_h, \psi_h) = \int_{I_j} I_h[(\phi_h, \psi_h)] - \frac{1}{6}h_j^2 \int_{I_j} (\phi_{hu}, \psi_{hu}) \tag{2.8}$$

for $j = 1, ..., N$ and all $\phi_h, \psi_h \in X_h^n$.
We use

$$Z_h := \left\{ z_h : [0, 2\pi] \to \mathbb{R}^n \mid z_{h|_{I_j}} \text{ is constant, } j = 1, ..., N \right\}, \tag{2.9}$$

and $Q_h : H^1((0, 2\pi), \mathbb{R}^n) \to Z_h$ given by

$$(Q_h f)_{|I_j} := \frac{1}{|I_j|} \int_{I_j} f, \; j = 1, .., N,$$

as introduced and defined in [22]. Then,

$$\|f - Q_h f\| \le Ch\|f\|_{H^1}, \quad \forall f \in H^1((0, 2\pi), \mathbb{R}^n). \tag{2.10}$$

For $\eta_h \in X_h^n$ and h satisfying (2.5) holds (see [21])

$$\|\eta_{hu}\|_{L^p} \leq Ch^{-1}\|\eta_h\|_{L^p}, \tag{2.11}$$

$$\|\eta_h\|_{L_\infty} \leq Ch^{-\frac{1}{2}}\|\eta_h\|. \tag{2.12}$$

2.3 Initial conditions

Here, we discuss a choice of the initial data and so bound the discrete and continuous problems together.
Due to the smoothness of the continuous solution there exist constants $0 < c_0 < C_0$ such that for $m = 0, ..., M$ and $T > 0$ holds

$$c_0 \leq |x_u^m| \leq C_0, \quad |y^m| \leq C_0 \quad \text{in } [0, 2\pi], \qquad \int_0^T \|y_u^m\|_{L^\infty}^2 \leq C_0.$$

We choose the following initial datum

$$x_h^0 = I_h x^0, \tag{2.13}$$

which determines the discrete curvature vector y_h^0 through the relation

$$\int_0^{2\pi} I_h\left[\left(y_h^0, \psi_h\right)\right]\left|\left(I_h x^0\right)_u\right| + \int_0^{2\pi} \frac{\left(\left(I_h x^0\right)_u, \psi_{hu}\right)}{\left|\left(I_h x^0\right)_u\right|} = 0 \tag{2.14}$$

for all $\psi_h \in X_h^n$. This is similar in spirit to [22]. The authors show that this choice of y_h^0 is a good approximation of the initial continuous problem.
Moreover, it can be shown that the following bounds are satisfied under the smallness assumption on the spatial grid size

$$\frac{1}{2}c_0 \leq |x_{hu}^0| \leq 2C_0, \quad |y_h^0| \leq 2C_0 \quad \text{in } [0, 2\pi]. \tag{2.15}$$

We show this for the upper bound on $|x_{hu}^0|$. From the interpolation estimate, triangle inequality and (2.13) one obtains

$$|x_{hu}^0| \leq \left|x_u^0\right| + \left|x_u^0 - x_{hu}^0\right| \leq C_0 + \left\|x_u^0 - \left(I_h x^0\right)_u\right\|_{L^\infty} \leq C_0 + Ch \leq 2C_0,$$

provided h is small enough. The remaining estimates will be discussed in detail in Section 3.1.

Chapter 3

Existence and uniqueness result. Second equation of the scheme

In this chapter, we shall be concerned with the unique solvability of the fully discrete scheme (1.11)-(1.12), proposed in Chapter 1. Using a suitable constrained minimization problem, we prove the existence of the solution at the next time step. To do so, we introduce a functional to be minimized and formulate a constrained minimization problem. The proof of the existence of the minimizer in the admissible set is presented in Lemma 3.2. In Lemma 3.3 we show that a minimizer of the given functional satisfies the equations (1.11)-(1.12), i.e. actually solves our problem. In order to motivate this proof we find the first variation of the minimized functional, what gives rise to the perturbation term of small order. The uniqueness result is obtained in Lemma 3.5 by taking the difference between two solutions and showing that this difference is equal to zero. We finish this chapter by discussing the second equation of our scheme and giving a different representation for it in the form of the equation (1.10) up to the remainder term R_h^{m+1}.

Before we formulate in Theorem 3.1 the main result of this chapter, we would like to draw a parallel between our approach to solving the problem with the one, proposed by Olischläger and M. Rumpf in [43]. The authors presented a nested variational time discretization with an inner minimization problem to be solved in each time step.

Theorem 3.1 (Existence and uniqueness). *Suppose* (x_h^m, y_h^m), $m \in [0, M-1]$ *satisfy*

$$\frac{1}{2}c_0 \le |x_{hu}^m| \le 2C_0, \quad |y_h^m| \le 2C_0 \quad \textit{in } [0, 2\pi], \tag{3.1}$$

where $c_0 > 0, C_0 > 0$ *are independent of* h *and* Δt *constants. Then there exist* $h_0 > 0$ *and* $0 < \mu \le 1$, *such that (1.11)-(1.12) has a unique solution* $\left(x_h^{m+1}, y_h^{m+1}\right)$, *satisfying*

$$\frac{1}{4}c_0 \le \left|x_{hu}^{m+1}\right| \le 4C_0, \quad \left|y_h^{m+1}\right| \le 4C_0 \quad \textit{in } [0, 2\pi], \tag{3.2}$$

for all $0 < h \le h_0$, *provided* $\Delta t \le \mu h^5$ *and the discrete energy is decreasing. The constants depend only on* c_0, C_0.

Here and throughout the thesis we will denote by C some positive generic constant that may vary from line to line and is independent of the time step size and mesh size. For practical reasons, we do not trace the exact value of the constant C, only indicating, where it is necessary, what this constant depends on.

3.1 Existence and uniqueness result

In this section we present the proof of Theorem 3.1.

3.1.1 Existence of the discrete solution

As already mentioned above, we first proof the existence of the discrete solution to (1.11)-(1.12) at the next time step. In the following, we formulate a minimization problem subject to a constraint and prove the existence of the minimizer in a certain class of admissible functions. To this purpose, we define for $(x_h, y_h) \in X_h^n \times X_h^n$ and $\forall \psi_h \in X_h^n$ the relation

$$\int_0^{2\pi} I_h\left[(y_h, \psi_h)\right]|x_{hu}| - \int_0^{2\pi} I_h\left[(y_h^m, \psi_h)\right]|x_{hu}^m| + \int_0^{2\pi} \frac{(P_h^m \psi_{hu}, x_{hu} - x_{hu}^m)}{|x_{hu}|} = 0 \qquad (3.3)$$

and introduce the admissible set

$$K := \left\{(x_h, y_h) \in X_h^n \times X_h^n | (x_h, y_h) \text{ satisfies (3.3) and } |x_{hu}| \geq \frac{1}{4}c_0\right\}. \qquad (3.4)$$

Let further $J : X_h^n \times X_h^n \to \mathbb{R}$ be the functional defined by

$$J(x_h, y_h) = \frac{1}{2}\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[|x_h - x_h^m|^2\right]|x_{hu}^m| + \frac{1}{2}\int_0^{2\pi} I_h\left[|y_h|^2\right]|x_{hu}| + \lambda\int_0^{2\pi}|x_{hu}|. \qquad (3.5)$$

Lemma 3.2 (Existence of the minimizer of the functional)**.** *The minimization problem*

$$\underset{(x_h,y_h)\in K}{min} J(x_h, y_h) \qquad (3.6)$$

has at least one solution $(\bar{x}_h, \bar{y}_h) \in K$. Moreover, there exist $h_0 > 0$ and $0 < \mu \leq 1$, such that the following bounds hold

$$\frac{1}{4}c_0 < |\bar{x}_{hu}| < 4C_0, \quad |\bar{y}_h| < 4C_0 \quad \textit{in } [0, 2\pi] \qquad (3.7)$$

for all $0 < h \leq h_0$, provided that $\Delta t \leq \mu h^5$. The constants depend only on c_0, C_0.

Proof. Let us first show that $K \neq \emptyset$. Indeed, $|x_{hu}^m| \geq \frac{1}{2}c_0 > \frac{1}{4}c_0$ and (x_h^m, y_h^m) belongs to $X_h^n \times X_h^n$ and satisfies (3.3), i.e.

$$\int_0^{2\pi} I_h\left[(y_h^m, \psi_h)\right]|x_{hu}^m| - \int_0^{2\pi} I_h\left[(y_h^m, \psi_h)\right]|x_{hu}^m| + \int_0^{2\pi} \frac{(P_h^m \psi_{hu}, x_{hu}^m - x_{hu}^m)}{|x_{hu}^m|} = 0.$$

Since $J \geq 0$, there exist $\alpha = \inf\limits_K J$ and a minimizing sequence $\left(x_h^j, y_h^j\right)_{j\in\mathbb{N}} \in K$ such that $J\left(x_h^j, y_h^j\right) \to \inf\limits_K J$. Using the lower bound on $|x_{hu}^m|$ as well as (2.7) and the fact that the minimizing sequence is admissible we obtain

$$\begin{aligned} C \geq& J\left(x_h^j, y_h^j\right) \geq \frac{1}{2}\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[\left|x_h^j - x_h^m\right|^2\right]|x_{hu}^m| + \frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^j\right|^2\right]|x_{hu}^j| \\ \geq& \frac{c_0}{4}\left(\frac{1}{2}\frac{1}{\Delta t}\left\|x_h^j - x_h^m\right\|^2 + \frac{1}{2}\left\|y_h^j\right\|^2\right). \end{aligned}$$

This result together with the equivalence of the norms in a finite-dimensional space implies the boundedness of $\left(x_h^j, y_h^j\right)_{j\in\mathbb{N}}$. The Bolzano-Weierstrass theorem applies and yields the existence of a convergent subsequence $\left(x_h^{j_k}, y_h^{j_k}\right)_{j\in\mathbb{N},k\in\mathbb{N}}$, with limit $(\bar{x}_h, \bar{y}_h) \in X_h^n \times X_h^n$, $(\bar{x}_h, \bar{y}_h) \in K$. From what follows $J\left(x_h^{j_k}, y_h^{j_k}\right) \to J(\bar{x}_h, \bar{y}_h) = \alpha$.
It remains to prove the bounds (3.7). Observing

$$J(\bar{x}_h, \bar{y}_h) \leq J(x_h^m, y_h^m) = \frac{1}{2}\int_0^{2\pi} I_h\left[|y_h^m|^2\right]|x_{hu}^m| + \lambda\int_0^{2\pi}|x_{hu}^m| \leq c(C_0)$$

and using (2.7) we obtain

$$c \geq \frac{1}{2}\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[|\bar{x}_h - x_h^m|^2\right]|x_{hu}^m| \geq \frac{c_0}{4}\frac{1}{\Delta t}\int_0^{2\pi}|\bar{x}_h - x_h^m|^2 = \frac{c_0}{4}\frac{1}{\Delta t}\left\|\bar{x}_h - x_h^m\right\|^2. \tag{3.8}$$

It follows now from (2.11) and (2.12) that

$$\left\|\bar{x}_{hu} - x_{hu}^m\right\|_{L^\infty} \leq Ch^{-1}\left\|\bar{x}_h - x_h^m\right\|_{L^\infty} \leq Ch^{-\frac{3}{2}}\left\|\bar{x}_h - x_h^m\right\|_{L^2} \leq Ch^{-\frac{3}{2}}\sqrt{\Delta t}, \tag{3.9}$$

where the constant C depends on c_0, C_0.
Hence, in view of (3.1) and using the condition $\Delta t \leq \mu h^5$ and $0 < \mu \leq 1$, we obtain

$$\begin{aligned} |\bar{x}_{hu}| =& |\bar{x}_{hu} - x_{hu}^m + x_{hu}^m| \leq |\bar{x}_{hu} - x_{hu}^m| + |x_{hu}^m| \leq \left\|\bar{x}_{hu} - x_{hu}^m\right\|_{L^\infty} + |x_{hu}^m| \\ \leq& C\sqrt{\Delta t}h^{-\frac{3}{2}} + 2C_0 \leq C\sqrt{\mu}h + 2C_0 \leq Ch + 2C_0 < 4C_0, \end{aligned} \tag{3.10}$$

provided h_0 is sufficiently small. Analogously, we have

$$\begin{aligned} |\bar{x}_{hu}| \geq& |x_{hu}^m| - \left\|\bar{x}_{hu} - x_{hu}^m\right\|_{L^\infty} \geq \frac{1}{2}c_0 - C\sqrt{\Delta t}h^{-\frac{3}{2}} \\ \geq& \frac{1}{2}c_0 - C\sqrt{\mu}h \geq \frac{1}{2}c_0 - Ch > \frac{1}{4}c_0. \end{aligned} \tag{3.11}$$

This proves the bounds on $|\bar{x}_{hu}|$ in (3.7). Let us consider the side constraint in order to get an upper bound on $|\bar{y}_h|$. Rearranging the terms and inserting the test function $\psi_h = \bar{y}_h - y_h^m$ into (3.3) we obtain

$$\begin{aligned} \int_0^{2\pi} I_h\left[|\bar{y}_h - y_h^m|^2\right]|\bar{x}_{hu}| =& \int_0^{2\pi} I_h\left[(y_h^m, \bar{y}_h - y_h^m)\right](|x_{hu}^m| - |\bar{x}_{hu}|) \\ &- \int_0^{2\pi}\frac{\left(P_h^m\left(\bar{x}_{hu} - x_{hu}^m\right), \bar{y}_{hu} - y_{hu}^m\right)}{|\bar{x}_{hu}|}. \end{aligned} \tag{3.12}$$

First, we estimate from above the terms on the right-hand side of the equation (3.12). Thus, Cauchy-Schwarz and Young's inequalities along with (2.7) and (3.9) provide

$$\begin{aligned}\left|\int_0^{2\pi} I_h\left[\left(y_h^m, \bar{y}_h - y_h^m\right)\right]\left(\left|x_{hu}^m\right| - \left|\bar{x}_{hu}\right|\right)\right| \leq & \int_0^{2\pi} I_h\left[\varepsilon\left|\bar{y}_h - y_h^m\right|^2 + C_\varepsilon\left|y_h^m\right|^2\right]\left|\bar{x}_{hu} - x_{hu}^m\right| \\ \leq & \left\|\bar{x}_{hu} - x_{hu}^m\right\|_{L^\infty}\left(\varepsilon C\left\|\bar{y}_h - y_h^m\right\|^2 + CC_\varepsilon\left\|y_h^m\right\|^2\right) \\ \leq & \varepsilon C\sqrt{\Delta t}h^{-\frac{3}{2}}\left\|\bar{y}_h - y_h^m\right\|^2 + CC_\varepsilon\sqrt{\Delta t}h^{-\frac{3}{2}}.\end{aligned}$$

Using the inverse estimate (2.11), Young's inequality, (3.8) as well as a lower bound (3.11) on $|\bar{x}_{hu}|$, obtained so far, we deduce

$$\begin{aligned}\left|\int_0^{2\pi}\frac{\left(P_h^m\left(\bar{x}_{hu} - x_{hu}^m\right), \bar{y}_{hu} - y_{hu}^m\right)}{\left|\bar{x}_{hu}\right|}\right| \leq & C\left\|\bar{y}_{hu} - y_{hu}^m\right\|\left\|\bar{x}_{hu} - x_{hu}^m\right\| \\ \leq & Ch^{-1}\left\|\bar{y}_h - y_h^m\right\|\left\|\bar{x}_{hu} - x_{hu}^m\right\| \\ \leq & \varepsilon\left\|\bar{y}_h - y_h^m\right\|^2 + CC_\varepsilon\Delta t h^{-4}.\end{aligned}$$

Let us next estimate from below the left-hand side of (3.12). From (2.7), (3.11) we infer

$$\int_0^{2\pi} I_h\left[\left|\bar{y}_h - y_h^m\right|^2\right]\left|\bar{x}_{hu}\right| \geq \frac{1}{4}c_0\left\|\bar{y}_h - y_h^m\right\|^2.$$

Combining three above estimates together and using the condition $\Delta t \leq \mu h^5$ we get

$$\begin{aligned}\frac{1}{4}c_0\left\|\bar{y}_h - y_h^m\right\|^2 \leq & \varepsilon\left(1 + C\sqrt{\Delta t}h^{-\frac{3}{2}}\right)\left\|\bar{y}_h - y_h^m\right\|^2 + CC_\varepsilon\left(\sqrt{\Delta t}h^{-\frac{3}{2}} + \Delta t h^{-4}\right) \\ \leq & \varepsilon\left(1 + C\sqrt{\mu}h\right)\left\|\bar{y}_h - y_h^m\right\|^2 + CC_\varepsilon\left(\sqrt{\mu}h + \mu h\right) \\ \leq & \varepsilon\left(1 + Ch\right)\left\|\bar{y}_h - y_h^m\right\|^2 + 2CC_\varepsilon\sqrt{\mu}h,\end{aligned}$$

where the last step follows from $0 < \mu \leq 1$. Choosing $\varepsilon = \frac{1}{8}c_0$ one obtains

$$\begin{aligned}&\frac{1}{8}c_0\left\|\bar{y}_h - y_h^m\right\|^2(1 - Ch) \leq \frac{4C}{c_0}\sqrt{\mu}h, \\ &\frac{1}{16}c_0\left\|\bar{y}_h - y_h^m\right\|^2 \leq \frac{4C}{c_0}\sqrt{\mu}h,\end{aligned}$$

provided h_0 is mall enough. The above estimate and an inverse inequality (2.11) imply

$$\left\|\bar{y}_h - y_h^m\right\|_{L^\infty}^2 \leq Ch^{-1}\left\|\bar{y}_h - y_h^m\right\|^2 \leq \frac{64C^2}{c_0^2}h^{-1}\sqrt{\mu}h \leq \frac{64C^2}{c_0^2}\sqrt{\mu} \leq C_0^2,$$

if $\sqrt{\mu} \leq \frac{C_0^2c_0^2}{64C^2}$. Setting $\sqrt{\mu} = \min\left\{1, \frac{C_0^2c_0^2}{64C^2}\right\}$, the next estimate completes the proof

$$\left|\bar{y}_h\right| \leq \left|\bar{y}_h - y_h^m\right| + \left|y_h^m\right| \leq \left\|\bar{y}_h - y_h^m\right\|_{L^\infty} + \left|y_h^m\right| \leq C_0 + 2C_0 < 4C_0.$$

□

Let us next show that the minimizer $(\bar{x}_h, \bar{y}_h) \in K$ of the functional J is a solution to our problem at the next time step.

Lemma 3.3. *Let $(\bar{x}_h, \bar{y}_h)$ be a solution of the constraint minimization problem (3.5)-(3.6), such that (3.7) holds. Then $(\bar{x}_h, \bar{y}_h)$ satisfies the equations*

$$\begin{aligned}\int_0^{2\pi} I_h\left[\left(\frac{\bar{x}_h - x_h^m}{\Delta t}, \phi_h\right)\right] |x_{hu}^m| - \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](\bar{\tau}_h, \phi_{hu}) - \int_0^{2\pi} \frac{(P_h^m \bar{y}_{hu}, \phi_{hu})}{|\bar{x}_{hu}|} \\ + \int_0^{2\pi} \frac{(P_h^m \bar{y}_{hu}, \bar{x}_{hu} - x_{hu}^m)}{|\bar{x}_{hu}|^3}(\bar{x}_{hu}, \phi_{hu}) + \lambda \int_0^{2\pi} (\bar{\tau}_h, \phi_{hu}) = 0.\end{aligned} \tag{3.13}$$

$$\int_0^{2\pi} I_h\left[(\bar{y}_h, \psi_h)\right] |\bar{x}_{hu}| - \int_0^{2\pi} I_h\left[(y_h^m, \psi_h)\right] |x_{hu}^m| + \int_0^{2\pi} \frac{(P_h^m \psi_{hu}, \bar{x}_{hu} - x_{hu}^m)}{|\bar{x}_{hu}|} = 0. \tag{3.14}$$

Proof. To begin, we note that $(\bar{x}_h, \bar{y}_h) \in K$ and therefore (3.14) is satisfied due to the definition (3.4) of K. Let us next for $\phi_h \in X_h^n$ and $\varepsilon > 0$ define

$$x_{\varepsilon,h} := \bar{x}_h + \varepsilon \phi_h, \quad y_{\varepsilon,h} := G(x_{\varepsilon,h}),$$

where $y_{\varepsilon,h}$ is uniquely determined through the relation

$$\int_0^{2\pi} I_h\left[(y_{\varepsilon,h}, \psi_h)\right] |x_{\varepsilon,hu}| - \int_0^{2\pi} I_h\left[(y_h^m, \psi_h)\right] |x_{hu}^m| + \int_0^{2\pi} \frac{(P_h^m \psi_{hu}, x_{\varepsilon,hu} - x_{hu}^m)}{|x_{\varepsilon,hu}|} = 0$$

for all $\psi_h \in X_h^n$ and small ε. We point out that the above relation represents a system of linear equations for $y_{\varepsilon,h}$, which is uniquely solvable, provided $|x_{\varepsilon,hu}|$ is bounded away from zero (what again follows from (3.4)).
Since the lower bound on $|\bar{x}_{hu}|$ in (3.7) is satisfied strictly, we can slightly perturb $(\bar{x}_h, \bar{y}_h)$ in the direction ϕ_h still staying in K. Therefore, $(x_{\varepsilon,h}, y_{\varepsilon,h}) \in K$ for sufficiently small ε.
Now let

$$\begin{aligned}f(\varepsilon) := J(x_{\varepsilon,h}, y_{\varepsilon,h}) =& \frac{1}{2}\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[|x_{\varepsilon,h} - x_h^m|^2\right] |x_{hu}^m| + \frac{1}{2}\int_0^{2\pi} I_h\left[|y_{\varepsilon,h}|^2\right] |x_{\varepsilon,hu}| \\ &+ \lambda \int_0^{2\pi} |x_{\varepsilon,hu}|.\end{aligned}$$

Since $(\bar{x}_h, \bar{y}_h)$ is a minimizer of J, $f(\varepsilon)$ must have a minimum at $\varepsilon = 0$, Therefore, $f'(0) = 0$.
Observing the following calculations

$$\begin{aligned}\frac{d}{d\varepsilon}(|\bar{x}_{hu} + \varepsilon\phi_{hu}|)|_{\varepsilon=0} =& \frac{d}{d\varepsilon}\left(\sqrt{(\bar{x}_{hu} + \varepsilon\phi_{hu}, \bar{x}_{hu} + \varepsilon\phi_{hu})}\right)\Big|_{\varepsilon=0} \\ =& \frac{\frac{d}{d\varepsilon}(\bar{x}_{hu} + \varepsilon\phi_{hu}, \bar{x}_{hu} + \varepsilon\phi_{hu})}{2\sqrt{(\bar{x}_{hu} + \varepsilon\phi_{hu}, \bar{x}_{hu} + \varepsilon\phi_{hu})}}\Big|_{\varepsilon=0} = \frac{2(\bar{x}_{hu}, \phi_{hu})}{2|\bar{x}_{hu}|} = (\bar{\tau}_h, \phi_{hu})\end{aligned}$$

one obtains

$$f'(\varepsilon)_{|\varepsilon=0} = \frac{d}{d\varepsilon} J\left(x_{\varepsilon,h}, y_{\varepsilon,h}\right)_{|\varepsilon=0} = \int_0^{2\pi} I_h\left[\left(\frac{\bar{x}_h - x_h^m}{\Delta t}, \phi_h\right)\right] |x_{hu}^m| + \int_0^{2\pi} I_h\left[(\bar{y}_h, y_{\phi,h})\right] |\bar{x}_{hu}| \\ + \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](\bar{\tau}_h, \phi_{hu}) + \lambda \int_0^{2\pi} (\bar{\tau}_h, \phi_{hu}).$$

Here we have introduced a notation $y_{\phi,h} = \frac{d}{d\varepsilon} G(x_{\varepsilon,h})_{|\varepsilon=0}$.
In order to find $y_{\phi,h}$, we consider the relation between x and its curvature vector y. To this purpose, for $\varepsilon > 0$ and $\psi_h \in X_h^n$ we define

$$g(x_{\varepsilon,h}, y_{\varepsilon,h}) = \int_0^{2\pi} I_h\left[(y_{\varepsilon,h}, \psi_h)\right] |x_{\varepsilon,h}| - \int_0^{2\pi} I_h\left[(y_h^m, \psi_h)\right] |x_{hu}^m| + \int_0^{2\pi} \frac{(P_h^m \psi_{hu}, x_{\varepsilon,h} - x_{hu}^m)}{|x_{\varepsilon,h}|}.$$

By definition of $y_{\varepsilon,h}$ we have $g(x_{\varepsilon,h}, y_{\varepsilon,h}) = 0$ and hence

$$0 = \frac{d}{d\varepsilon} g(x_{\varepsilon,h}, y_{\varepsilon,h})_{|\varepsilon=0} = \int_0^{2\pi} I_h\left[(y_{\phi,h}, \psi_h)\right] |\bar{x}_{hu}| + \int_0^{2\pi} I_h\left[(y_h, \psi_h)\right](\bar{\tau}_h, \phi_{hu}) \\ + \int_0^{2\pi} \frac{(P_h^m \psi_{hu}, \phi_{hu})}{|\bar{x}_{hu}|} - \int_0^{2\pi} \frac{(P_h^m \psi_{hu}, \bar{x}_{hu} - x_{hu}^m)}{|\bar{x}_{hu}|^3}(\bar{x}_{hu}, \phi_{hu}).$$

Inserting $\psi_h = \bar{y}_h$ into the above equation we deduce

$$\int_0^{2\pi} I_h\left[(y_{\phi,h}, \bar{y}_h)\right] |\bar{x}_{hu}| = -\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](\bar{\tau}_h, \phi_{hu}) - \int_0^{2\pi} \frac{(P_h^m \bar{y}_{hu}, \phi_{hu})}{|\bar{x}_{hu}|} \\ + \int_0^{2\pi} \frac{(P_h^m \bar{y}_{hu}, \bar{x}_{hu} - x_{hu}^m)}{|\bar{x}_{hu}|^3}(\bar{x}_{hu}, \phi_{hu}).$$

Using the above representation we are able to rewrite the first derivative of f at $\varepsilon = 0$ as

$$0 = f'(0) = \int_0^{2\pi} I_h\left[\left(\frac{\bar{x}_h - x_h^m}{\Delta t}, \phi_h\right)\right] |\bar{x}_{hu}^m| - \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](\bar{\tau}_h, \phi_{hu}) \\ - \int_0^{2\pi} \frac{(P_h^m \bar{y}_{hu}, \phi_{hu})}{|\bar{x}_{hu}|} + \int_0^{2\pi} \frac{(P_h^m \bar{y}_{hu}, \bar{x}_{hu} - x_{hu}^m)}{|\bar{x}_{hu}|^3}(\bar{x}_{hu}, \phi_{hu}) + \lambda \int_0^{2\pi} (\bar{\tau}_h, \phi_{hu}),$$

which proves our assertion above. □

Remark 3.4. *We note that to prove the existence it is essential that the lower bound on $|\bar{x}_{hu}|$ is strictly satisfied. From (3.11) follows that this can be achieved under the condition $\Delta t \leq \varepsilon h^3$, where $\varepsilon = \varepsilon(c_0, C_0)$ is sufficiently small positive constant. Therefore, the existence result can be obtained under the milder condition $\left(\Delta t \leq \varepsilon h^3\right)$, whereas the stronger condition $\Delta t < \mu h^5$ provides the uniform control on the curvature vector, which is crucial for the error analysis.*

3.1.2 Uniqueness of the discrete solution

Lemma 3.5 (Uniqueness of the discrete solution)**.** *There exist $h_0 > 0$ and $0 < \mu^* \leq 1$, such that the system of equations (1.11)-(1.12) has a unique solution (x_h^*, y_h^*), provided*

$$\frac{1}{4}c_0 \leq |x_{hu}^*| \leq 4C_0, \quad |y_h^*| \leq 4C_0 \quad \textit{in } [0, 2\pi] \tag{3.15}$$

and $\Delta t \leq \mu^* h^4$.

Proof. Let us assume that $(\bar{x}_h, \bar{y}_h)$, $(\hat{x}_h, \hat{y}_h) \in X_h^n \times X_h^n$ are two solutions of the equations (1.11)-(1.12) satisfying (3.15). Our aim is to show that they identically coincide. Taking the difference between the corresponding equations for $(\bar{x}_h, \bar{y}_h)$ and $(\hat{x}_h, \hat{y}_h)$ we obtain

$$\begin{aligned}
&\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[(\bar{x}_h - \hat{x}_h, \phi_h)\right]|x_{hu}^m| - \int_0^{2\pi}\left(P_h^m\left(\frac{\bar{y}_{hu}}{|\bar{x}_{hu}|} - \frac{\hat{y}_{hu}}{|\hat{x}_{hu}|}\right), \phi_{hu}\right) \\
&- \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](\bar{\tau}_h, \phi_{hu}) + \frac{1}{2}\int_0^{2\pi} I_h\left[|\hat{y}_h|^2\right](\hat{\tau}_h, \phi_{hu}) \\
&+ \int_0^{2\pi}\frac{(P_h^m \bar{y}_{hu}, \bar{x}_{hu})}{|\bar{x}_{hu}|^3}(\bar{x}_{hu}, \phi_{hu}) - \int_0^{2\pi}\frac{(P_h^m \hat{y}_{hu}, \hat{x}_{hu})}{|\hat{x}_{hu}|^3}(\hat{x}_{hu}, \phi_{hu}) \\
&\qquad\qquad\qquad\qquad + \lambda\int_0^{2\pi}(\bar{\tau}_h - \hat{\tau}_h, \phi_{hu}) = 0,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\int_0^{2\pi} I_h\left[(\bar{y}_h, \psi_h)\right]|\bar{x}_{hu}| - \int_0^{2\pi} I_h\left[(\hat{y}_h, \psi_h)\right]|\hat{x}_{hu}| \\
&\qquad\qquad + \int_0^{2\pi}\frac{(P_h^m \bar{x}_{hu}, \psi_{hu})}{|\bar{x}_{hu}|} - \int_0^{2\pi}\frac{(P_h^m \hat{x}_{hu}, \psi_{hu})}{|\hat{x}_{hu}|} = 0.
\end{aligned} \tag{3.17}$$

Here we used the definition of the projection matrix $P_h^m = I_n - \tau_h^m \otimes \tau_h^m$ and the fact that $P_h^m \tau_h^m = 0$. The first two terms in (3.17) can be rewritten as

$$\begin{aligned}
&\int_0^{2\pi} I_h\left[(\bar{y}_h, \psi_h)\right]|\bar{x}_{hu}| - \int_0^{2\pi} I_h\left[(\hat{y}_h, \psi_h)\right]|\hat{x}_{hu}| \\
&= \int_0^{2\pi} I_h\left[(\bar{y}_h - \hat{y}_h, \psi_h)\right]|\bar{x}_{hu}| - \int_0^{2\pi} I_h\left[(\hat{y}_h, \psi_h)\right](|\hat{x}_{hu}| - |\bar{x}_{hu}|).
\end{aligned}$$

We insert the following test functions $\phi_h = \bar{x}_h - \hat{x}_h$ and $\psi_h = \bar{y}_h - \hat{y}_h$ into (3.16) and (3.17), respectively, and obtain the following equations using the above representation

$$\begin{aligned}
&\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[|\bar{x}_h - \hat{x}_h|^2\right]|x_{hu}^m| - \int_0^{2\pi}\left(P_h^m\left(\frac{\bar{y}_{hu}}{|\bar{x}_{hu}|} - \frac{\hat{y}_{hu}}{|\hat{x}_{hu}|}\right), \bar{x}_{hu} - \hat{x}_{hu}\right) \\
&- \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](\bar{\tau}_h, \bar{x}_{hu} - \hat{x}_{hu}) + \frac{1}{2}\int_0^{2\pi} I_h\left[|\hat{y}_h|^2\right](\hat{\tau}_h, \bar{x}_{hu} - \hat{x}_{hu}) \\
&+ \int_0^{2\pi}\frac{(P_h^m \bar{y}_{hu}, \bar{x}_{hu})}{|\bar{x}_{hu}|^3}(\bar{x}_{hu}, \bar{x}_{hu} - \hat{x}_{hu}) - \int_0^{2\pi}\frac{(P_h^m \hat{y}_{hu}, \hat{x}_{hu})}{|\hat{x}_{hu}|^3}(\hat{x}_{hu}, \bar{x}_{hu} - \hat{x}_{hu}) \\
&\qquad\qquad\qquad\qquad + \lambda\int_0^{2\pi}(\bar{\tau}_h - \hat{\tau}_h, \bar{x}_{hu} - \hat{x}_{hu}) = 0,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}\int_0^{2\pi} I_h\left[|\bar{y}_h-\hat{y}_h|^2\right]|\bar{x}_{hu}| - \int_0^{2\pi} I_h\left[(\hat{y}_h,\bar{y}_h-\hat{y}_h)\right](|\hat{x}_{hu}|-|\bar{x}_{hu}|)\\ + \int_0^{2\pi}\frac{(P_h^m\bar{x}_{hu},\bar{y}_{hu}-\hat{y}_{hu})}{|\bar{x}_{hu}|} - \int_0^{2\pi}\frac{(P_h^m\hat{x}_{hu},\bar{y}_{hu}-\hat{y}_{hu})}{|\hat{x}_{hu}|} = 0.\end{aligned} \tag{3.19}$$

As the next step we sum two equations (3.18), (3.19) and after rearranging achieve

$$\begin{aligned}&\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[|\bar{x}_h-\hat{x}_h|^2\right]|x_{hu}^m| + \int_0^{2\pi} I_h\left[|\bar{y}_h-\hat{y}_h|^2\right]|\bar{x}_{hu}|\\
&= \int_0^{2\pi} I_h\left[(\hat{y}_h,\bar{y}_h-\hat{y}_h)\right](|\hat{x}_{hu}|-|\bar{x}_{hu}|) + \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](\bar{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu})\\
&\quad - \frac{1}{2}\int_0^{2\pi} I_h\left[|\hat{y}_h|^2\right](\hat{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu}) + \int_0^{2\pi}\left(P_h^m\left(\frac{\bar{y}_{hu}}{|\bar{x}_{hu}|}-\frac{\hat{y}_{hu}}{|\hat{x}_{hu}|}\right),\bar{x}_{hu}-\hat{x}_{hu}\right)\\
&\quad - \int_0^{2\pi}\frac{(P_h^m\bar{x}_{hu},\bar{y}_{hu}-\hat{y}_{hu})}{|\bar{x}_{hu}|} + \int_0^{2\pi}\frac{(P_h^m\hat{x}_{hu},\bar{y}_{hu}-\hat{y}_{hu})}{|\hat{x}_{hu}|}\\
&\quad - \int_0^{2\pi}\frac{(P_h^m\bar{y}_{hu},\bar{x}_{hu})}{|\bar{x}_{hu}|^3}(\bar{x}_{hu},\bar{x}_{hu}-\hat{x}_{hu}) + \int_0^{2\pi}\frac{(P_h^m\hat{y}_{hu},\hat{x}_{hu})}{|\hat{x}_{hu}|^3}(\hat{x}_{hu},\bar{x}_{hu}-\hat{x}_{hu})\\
&\quad - \lambda\int_0^{2\pi}(\bar{\tau}_h-\hat{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu})\\
&= \sum_{i=1}^{9} S_i.\end{aligned} \tag{3.20}$$

Let us estimate the left- and right-hand sides of (3.20) in order to show that the difference between two solutions is zero, provided Δt is small enough. We start with the terms S_1, S_2 and S_3. First, we complete the square for S_1 and rewrite S_2 as well as S_3 using the definition of the tangent vector. This results in

$$\begin{aligned}S_1+S_2+S_3 = &-\frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h-\hat{y}_h|^2\right](|\hat{x}_{hu}|-|\bar{x}_{hu}|) + \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](|\hat{x}_{hu}|-|\bar{x}_{hu}|)\\
&-\frac{1}{2}\int_0^{2\pi} I_h\left[|\hat{y}_h|^2\right](|\hat{x}_{hu}|-|\bar{x}_{hu}|) + \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right]|\bar{x}_{hu}|\\
&-\frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right](\bar{\tau}_h,\hat{\tau}_h)|\hat{x}_{hu}| - \frac{1}{2}\int_0^{2\pi} I_h\left[|\hat{y}_h|^2\right](\hat{\tau}_h,\bar{\tau}_h)|\bar{x}_{hu}|\\
&+\frac{1}{2}\int_0^{2\pi} I_h\left[|\hat{y}_h|^2\right]|\hat{x}_{hu}|.\end{aligned}$$

Using the following relation

$$(v,w) = 1-\frac{1}{2}|v-w|^2 \quad \text{for } v,w\in\mathbb{R}^n, |v|=|w|=1, \tag{3.21}$$

simplifying and rearranging the terms we arrive at

$$\begin{aligned} S_1 + S_2 + S_3 =& \frac{1}{4}\int_0^{2\pi} I_h\left[|\bar{y}_h|^2\right]|\bar{\tau}_h - \hat{\tau}_h|^2 |\hat{x}_{hu}| + \frac{1}{4}\int_0^{2\pi} I_h\left[|\hat{y}_h|^2\right]|\hat{\tau}_h - \bar{\tau}_h|^2 |\bar{x}_{hu}| \\ & - \frac{1}{2}\int_0^{2\pi} I_h\left[|\bar{y}_h - \hat{y}_h|^2\right](|\hat{x}_{hu}| - |\bar{x}_{hu}|) \\ =& I + II + III. \end{aligned} \tag{3.22}$$

In order to estimate the first two terms on the right-hand side of (3.22), we have to deal with the difference of the tangent vectors. To this purpose, we first derive with the help of the triangle and reverse triangle inequalities the following estimate

$$\begin{aligned} \left|\frac{a}{|a|} - \frac{b}{|b|}\right| &= \left|\frac{a|b| - b|a| \mp a|a|}{|a||b|}\right| = \left|\frac{a\,(|b| - |a|) + (a-b)\,|a|}{|a||b|}\right| \\ &\le \frac{|a|\,||b| - |a||}{|a||b|} + \frac{|a-b|\,|a|}{|a||b|} \le \frac{|b-a|}{|b|} + \frac{|a-b|}{|b|} \le 2\frac{|a-b|}{|b|}. \end{aligned} \tag{3.23}$$

Recalling now the definition of the unit tangent vector and setting $a = \bar{x}_{hu}$ and $b = \hat{x}_{hu}$ in the above inequality we obtain in view of (3.15)

$$|\bar{\tau}_h - \hat{\tau}_h| = \left|\frac{\bar{x}_{hu}}{|\bar{x}_{hu}|} - \frac{\hat{x}_{hu}}{|\hat{x}_{hu}|}\right| \le 2\frac{|\bar{x}_{hu} - \hat{x}_{hu}|}{|\hat{x}_{hu}|} \le C\,|\bar{x}_{hu} - \hat{x}_{hu}|\,. \tag{3.24}$$

Using (2.7), (3.15) and the above calculations we deduce

$$I \le C\int_0^{2\pi} |\bar{y}_h|^2 |\bar{\tau}_h - \hat{\tau}_h|^2 \le C\,\|\bar{x}_{hu} - \hat{x}_{hu}\|^2 \le Ch^{-2}\,\|\hat{x}_h - \bar{x}_h\|^2\,.$$

The second term II on the right-hand side of (3.22) can be estimated in the same way. In view of (2.7), (3.15), reverse triangle and Young's inequalities we get

$$\begin{aligned} III &\le C\int_0^{2\pi} (|\bar{y}_h| + |\hat{y}_h|)\,|\bar{y}_h - \hat{y}_h|\,|\hat{x}_{hu} - \bar{x}_{hu}| \le Ch^{-1}\,\|\bar{y}_h - \hat{y}_h\|\,\|\hat{x}_h - \bar{x}_h\| \\ &\le \varepsilon\,\|\bar{y}_h - \hat{y}_h\|^2 + C_\varepsilon h^{-2}\,\|\hat{x}_h - \bar{x}_h\|^2\,. \end{aligned}$$

Combining S_4, S_5 and S_6 we obtain after simplifying the terms

$$\begin{aligned} S_4 + S_5 + S_6 =& \int_0^{2\pi} (P_h^m \hat{y}_{hu}, \bar{x}_{hu} - \hat{x}_{hu})\left(\frac{1}{|\bar{x}_{hu}|} - \frac{1}{|\hat{x}_{hu}|}\right) \\ & + \int_0^{2\pi} (P_h^m(\hat{y}_{hu} - \bar{y}_{hu}), \hat{x}_{hu})\left(\frac{1}{|\bar{x}_{hu}|} - \frac{1}{|\hat{x}_{hu}|}\right). \end{aligned}$$

With the help of (3.15) and an inverse inequality (2.11) we have

$$\begin{aligned} S_4 + S_5 + S_6 \le& C\,\|\bar{y}_{hu}\|_{L^\infty}\,\|\bar{x}_{hu} - \hat{x}_{hu}\|^2 + C\,\|\bar{y}_{hu} - \hat{y}_{hu}\|\,\|\bar{x}_{hu} - \hat{x}_{hu}\| \\ \le& Ch^{-3}\,\|\bar{x}_h - \hat{x}_h\|^2 + Ch^{-2}\,\|\bar{y}_h - \hat{y}_h\|\,\|\bar{x}_h - \hat{x}_h\| \\ \le& \varepsilon\,\|\bar{y}_h - \hat{y}_h\|^2 + \left(Ch^{-3} + C_\varepsilon h^{-4}\right)\|\bar{x}_h - \hat{x}_h\|^2\,. \end{aligned}$$

Next, we recall the definition of the tangent vector to rearrange the following two terms

$$\begin{aligned} S_7+S_8=&-\int_0^{2\pi}\frac{\left(P_h^m\bar{y}_{hu},\bar{\tau}_h\right)}{|\bar{x}_{hu}|}\left(\bar{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu}\right)+\int_0^{2\pi}\frac{\left(P_h^m\hat{y}_{hu},\hat{\tau}_h\right)}{|\hat{x}_{hu}|}\left(\hat{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu}\right)\\ =&\int_0^{2\pi}\frac{\left(P_h^m\hat{y}_{hu},\hat{\tau}_h\right)}{|\hat{x}_{hu}|}\left(\hat{\tau}_h-\bar{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu}\right)+\int_0^{2\pi}\frac{\left(P_h^m\hat{y}_{hu},\hat{\tau}_h-\bar{\tau}_h\right)}{|\hat{x}_{hu}|}\left(\bar{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu}\right)\\ &+\int_0^{2\pi}\frac{\left(P_h^m\left(\hat{y}_{hu}-\bar{y}_{hu}\right),\bar{\tau}_h\right)}{|\hat{x}_{hu}|}\left(\bar{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu}\right)\\ &+\int_0^{2\pi}\left(P_h^m\bar{y}_{hu},\bar{\tau}_h\right)\left(\bar{\tau}_h,\bar{x}_{hu}-\hat{x}_{hu}\right)\left(\frac{1}{|\hat{x}_{hu}|}-\frac{1}{|\bar{x}_{hu}|}\right). \end{aligned}$$

Now, the reverse triangle inequality along with (3.15), (3.24) and the inverse estimate (2.11) implies

$$\begin{aligned} S_7+S_8\le&C\left(\|\hat{y}_{hu}\|_{L^\infty}+\|\bar{y}_{hu}\|_{L^\infty}\right)\|\bar{x}_{hu}-\hat{x}_{hu}\|^2+C\|\bar{y}_{hu}-\hat{y}_{hu}\|\|\bar{x}_{hu}-\hat{x}_{hu}\|\\ \le&\varepsilon\|\bar{y}_h-\hat{y}_h\|^2+C_\varepsilon h^{-4}\|\bar{x}_h-\hat{x}_h\|^2. \end{aligned}$$

It remains to deal with the ninth term. Recalling (3.21) one derives

$$\begin{aligned} S_9=&-\lambda\int_0^{2\pi}\left(\bar{\tau}_h-\hat{\tau}_h,|\bar{x}_{hu}|\bar{\tau}_h-|\hat{x}_{hu}|\hat{\tau}_h\right)=-\frac{\lambda}{2}\int_0^{2\pi}\left(1-(\bar{\tau}_h,\hat{\tau}_h)\right)\left(|\bar{x}_{hu}|+|\hat{x}_{hu}|\right)\\ =&-\frac{\lambda}{2}\int_0^{2\pi}|\bar{\tau}_h-\hat{\tau}_h|^2\left(|\bar{x}_{hu}|+|\hat{x}_{hu}|\right)\le0. \end{aligned}$$

Finally, with the help of (2.6) we estimate from below the left-hand side of (3.20)

$$\int_0^{2\pi}I_h\left[\left|\frac{\bar{x}_h-\hat{x}_h}{\Delta t}\right|^2\right]|x_{hu}^m|+\int_0^{2\pi}I_h\left[|\bar{y}_h-\hat{y}_h|^2\right]|\bar{x}_{hu}|\ge\frac{c_0}{2}\frac{1}{\Delta t}\|\bar{x}_h-\hat{x}_h\|^2+\frac{c_0}{4}\|\bar{y}_h-\hat{y}_h\|^2.$$

Combining the calculations above we get as a result

$$\|\bar{x}_h-\hat{x}_h\|^2\left(\frac{c_0}{2}\frac{1}{\Delta t}-C_\varepsilon h^{-2}-2Ch^{-2}-Ch^{-3}-2C_\varepsilon h^{-4}\right)+\|\bar{y}_h-\hat{y}_h\|^2\left(\frac{c_0}{4}-3\varepsilon\right)\le0.$$

Choosing $\varepsilon=\dfrac{c_0}{16}$ and multiplying the above inequality by Δt we arrive at

$$\|\bar{x}_h-\hat{x}_h\|^2\left(\frac{c_0}{2}-\Delta t\left(\frac{4}{c_0}h^{-2}+Ch^{-2}+Ch^{-3}+\frac{8}{c_0}h^{-4}\right)\right)+\frac{c_0}{16}\Delta t\|\bar{y}_h-\hat{y}_h\|^2\le0.$$

Using the condition $\Delta t\le\mu^*h^4$ we estimate the coefficients in front of $\|\bar{x}_h-\hat{x}_h\|^2$

$$\begin{aligned} &\frac{c_0}{2}-\Delta th^{-4}\left(\frac{4}{c_0}h^2+Ch^2+Ch+\frac{8}{c_0}\right)\ge\frac{c_0}{2}-\mu^*\left(\frac{4}{c_0}h^2+Ch^2+Ch+\frac{8}{c_0}\right)\\ &\ge\frac{c_0}{2}-\left(\frac{4}{c_0}h^2+Ch^2+Ch\right)-\mu^*\frac{8}{c_0}\ge\frac{c_0}{2}-\frac{c_0}{8}-\mu^*\frac{8}{c_0}\ge\frac{3c_0}{8}-\mu^*\frac{8}{c_0}\ge\frac{c_0}{4}>0, \end{aligned}$$

provided $h\le h_0$ and $\mu^*\le\dfrac{c_0^2}{64}$. Let $\mu^*=\dfrac{c_0^2}{64}$ and the uniqueness of the discrete solution follows immediately from $c_0>0$. □

3.1.3 Proof of Theorem 3.1

Proof. The existence of the discrete solution $\left(x_h^{m+1}, y_h^{m+1}\right)$ of the system (1.11)-(1.12) follows from Lemma 3.2 and Lemma 3.3 by choosing $(\bar{x}_h, \bar{y}_h) =: \left(x_h^{m+1}, y_h^{m+1}\right)$. Then,

$$J\left(x_h^{m+1}, y_h^{m+1}\right) \leq J\left(x_h^m, y_h^m\right) = \frac{1}{2}\int_0^{2\pi} I_h\left[|y_h^m|^2\right]|x_{hu}^m| + \lambda\int_0^{2\pi}|x_{hu}^m|$$

and the decrease of the discrete energy is observed

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^{m+1}\right|^2\right]\left|x_{hu}^{m+1}\right| + \lambda\int_0^{2\pi}\left|x_{hu}^{m+1}\right|& \\ -\left(\frac{1}{2}\int_0^{2\pi} I_h\left[|y_h^m|^2\right]|x_{hu}^m| + \lambda\int_0^{2\pi}|x_{hu}^m|\right)& \\ \leq -\frac{1}{2}\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[\left|x_h^{m+1} - x_h^m\right|^2\right]|x_{hu}^m| \leq 0,&\end{aligned}$$

what results in the stability property of the fully discrete scheme (1.11)-(1.12). Moreover, from (3.7) we also deduce that the discrete solution satisfies bounds (3.2). Finally, the uniqueness result from Lemma 3.5 completes the proof. □

Remark 3.6. *It follows from (3.8) and (3.9) with* $\bar{x}_h = x_h^{m+1}$

$$\left\|x_{hu}^{m+1} - x_{hu}^m\right\|_{L^\infty} \leq Ch^{-\frac{3}{2}}\left\|x_h^{m+1} - x_h^m\right\|_{L^2} \leq C\sqrt{\Delta t}h^{-\frac{3}{2}}, \quad m = 0, ..., M-1. \tag{3.25}$$

3.2 Second equation of the scheme

The second equation (1.12) of our scheme is a discrete analogue of the equation (1.10) differentiated with respect to time. In our further analysis we will often require the discrete equivalent of the equation (1.10). Therefore, below we derive this representation up to the remainder term R_h^{m+1}.

Lemma 3.7 (Remainder term)**.** *Suppose (2.14) holds. Then for* $m = 1, ..., M-1$ *the equation (1.12) can be written in the form*

$$\int_0^{2\pi} I_h\left[\left(y_h^{m+1}, \psi_h\right)\right]\left|x_{hu}^{m+1}\right| + \int_0^{2\pi}\left(\tau_h^{m+1}, \psi_{hu}\right) + \left\langle R_h^{m+1}, \psi_h\right\rangle = 0, \tag{3.26}$$

where the remainder term given by

$$\left\langle R_h^{m+1}, \psi_h\right\rangle = \frac{1}{2}\sum_{k=0}^{m}\int_0^{2\pi}\left(\tau_h^k, \psi_{hu}\right)\left|\tau_h^{k+1} - \tau_h^k\right|^2. \tag{3.27}$$

Proof. From the equation (1.12) we have for $k = 0, ..., m$

$$\int_0^{2\pi} I_h\left[\left(y_h^{k+1}, \psi_h\right)\right]\left|x_{hu}^{k+1}\right| - \int_0^{2\pi} I_h\left[\left(y_h^k, \psi_h\right)\right]\left|x_{hu}^k\right| + \int_0^{2\pi} \frac{\left(P_h^k\left(x_{hu}^{k+1} - x_{hu}^k\right), \psi_{hu}\right)}{\left|x_{hu}^{k+1}\right|} = 0. \tag{3.28}$$

Let us rewrite the last term in (3.28) using the definition of the projection matrix $P_h^k = I_n - \tau_h^k \otimes \tau_h^k$ and the fact that $P_h^k \tau_h^k = 0$

$$\begin{aligned}\frac{\left(P_h^k\left(x_{hu}^{k+1} - x_{hu}^k\right), \psi_{hu}\right)}{\left|x_{hu}^{k+1}\right|} &= \left(\tau_h^{k+1}, \psi_{hu}\right) - \left(\tau_h^{k+1}, \tau_h^k\right)\left(\tau_h^k, \psi_{hu}\right) \mp \left(\tau_h^k, \psi_{hu}\right) \\ &= \left(\tau_h^{k+1}, \psi_{hu}\right) - \left(\tau_h^k, \psi_{hu}\right) + \frac{1}{2}\left(\tau_h^k, \psi_{hu}\right)\left|\tau_h^{k+1} - \tau_h^k\right|^2.\end{aligned}$$

Equation (3.28) now can be written as

$$\begin{aligned}\int_0^{2\pi} I_h\left[\left(y_h^{k+1}, \psi_h\right)\right]\left|x_{hu}^{k+1}\right| - \int_0^{2\pi} I_h\left[\left(y_h^k, \psi_h\right)\right]\left|x_{hu}^k\right| + \int_0^{2\pi}\left(\tau_h^{k+1}, \psi_{hu}\right) - \int_0^{2\pi}\left(\tau_h^k, \psi_{hu}\right) \\ + \frac{1}{2}\int_0^{2\pi}\left(\tau_h^k, \psi_{hu}\right)\left|\tau_h^{k+1} - \tau_h^k\right|^2 = 0.\end{aligned}$$

Summing up from $k = 0$ to $k = m$ and using (2.14) we obtain the claim of the lemma. □

Corollary 3.8 (Numerical scheme with the remainder term)**.** *Using Lemma 3.7 we are able to rewrite equations (1.11)-(1.12) of our scheme in the following way*

$$\begin{aligned}\int_0^{2\pi} I_h\left[\left(\frac{x_h^{m+1} - x_h^m}{\Delta t}, \phi_h\right)\right]\left|x_{hu}^m\right| - \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, \phi_{hu}\right)}{\left|x_{hu}^{m+1}\right|} \\ - \frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^{m+1}\right|^2\right]\left(\tau_h^{m+1}, \phi_{hu}\right) + \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1} - x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1}, \phi_{hu}\right) \\ + \lambda\int_0^{2\pi}\left(\tau_h^{m+1}, \phi_{hu}\right) = 0,\end{aligned} \tag{3.29}$$

$$\int_0^{2\pi} I_h\left[\left(y_h^{m+1}, \psi_h\right)\right]\left|x_{hu}^{m+1}\right| + \int_0^{2\pi}\left(\tau_h^{m+1}, \psi_{hu}\right) + \left\langle R_h^{m+1}, \psi_h\right\rangle = 0. \tag{3.30}$$

Chapter 4

Error analysis

In this Chapter, we present the main results of our work. We start by formulating the error estimates in terms of the L^2-norms in Theorem 4.1. To prove the error bounds, we set up in Theorem 4.2 the finite induction argument, which will be closed in Theorem 5.2 after the error analysis is finished. The basic idea is as follows: given a discrete solution at the previous time step, we carry out the error analysis and obtain the estimates for the desired norms at the next time step. Thus, in Lemma 4.4 and Lemma 4.5 we present the estimates for the errors between continuous and discrete position and curvature vectors, respectively. In Lemma 4.6 we combine the results of Lemma 4.4 and Lemma 4.5. On the right-hand side of the inequality obtained in this lemma we get the differences between continuous and discrete tangent and curvature vectors, which we treat in Lemma 4.7 and Lemma 4.10, respectively. We deal separately with the error in the length element, deriving first the expression for the discrete length element in Lemma 4.11. Finally, we estimate the norm $\left\| |x_u^{m+1}| - |x_{hu}^{m+1}| \right\|$ in Lemma 4.12 with the help of the discrete Gronwall's Lemma. This proof completes the chapter.

We emphasize that our work treats a fully discrete case and thus extends the results of the work [22] by Deckelnick and Dziuk, where a semidiscrete scheme to approximate the elastic flow of curves was analyzed. Therefore we follow the structure of the proof of that paper, in which the error analysis is carried out in a sequence of lemmas.

4.1 Error bounds. Induction argument

Throughout the thesis we will use the regularity of the solution to the continuous problem. To this intent, we formulate the following helpful result: There exist constants $0 < c_0 < C_0$ and $0 < c_1 < C_1$, such that for $m = 0, ..., M$ in $[0, 2\pi]$ holds:

$$c_0 \leq |x_u^m| \leq C_0, \quad |y^m| \leq C_0, \quad \sum_{k=0}^{m} \Delta t \left\| y_u^k \right\|_{L^\infty}^2 \leq C_0, \tag{4.1}$$

$$|x_t^m| \leq C_1, \quad |y_u^m| \leq C_1, \tag{4.2}$$

$$\frac{1}{2}c_0 \leq \left|\left(I_h x^m\right)_u\right| \leq \frac{3}{2}\,C_0, \tag{4.3}$$

$$\left|I_h x_t^m\right| \leq \frac{3}{2}\,C_1, \left|I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right]\right| \leq 2\,C_1, \tag{4.4}$$

$$\left|P^m y^m\right|, \left|P^m y_u^m\right|, \left|P^m y_{uu}^m\right|, \left|P_u^m y^m\right|, \left|P_u^m y_u^m\right| \leq C, \tag{4.5}$$

where the constant C depends only on the norms of the continuous solution. In view of the smoothness of the continuous solution and the extreme value theorem, the estimates (4.1) and (4.2) hold.
Let us consider $\left(I_h x^m\right)_u$ defined on $[0, 2\pi]$ and restricted on subinterval I_j for some $j \in \{1, .., N\}$. Then from the Taylor expansion we infer

$$\left|\left(I_h x^m\right)_{u|I_j}\right| = \left|\frac{x^m(u_j) - x^m(u_{j-1})}{h}\right| = \left|x_u^m\left(u_{j-1}\right) + o(1)\right| \leq \frac{3}{2}\,C_0,$$
$$\left|\left(I_h x^m\right)_{u|I_j}\right| = \left|\frac{x^m(u_j) - x^m(u_{j-1})}{h}\right| = \left|x_u^m\left(u_{j-1}\right) + o(1)\right| \geq \frac{1}{2}c_0,$$

provided h is small enough.
Recalling the definition of the projection matrix (1.6) as well as (4.1) we deduce

$$\left|P^m y^m\right| = \left|y^m - \left(\tau^m, y^m\right)\tau^m\right| \leq C_0.$$

Similarly, we get the remaining estimates.
It is convenient to decompose the error $e^m = x^m - x_h^m$ in the following way:

$$e^m = \left(x^m - I_h x^m\right) + \left(I_h x^m - x_h^m\right) =: \epsilon^m + e_h^m, \tag{4.6}$$

where $x^m = x(\cdot, m\Delta t)$ represents the solution of (1.2) evaluated at $m\Delta t$.
In the next theorem we formulate the main result of our work.

Theorem 4.1 (Error bounds)**.** *Let $x : [0, 2\pi] \times [0, T] \to \mathbb{R}^n$ be a sufficiently smooth solution of (1.2). Then there exist $h_0 > 0$, $\Delta t_0 > 0$ and $\mu > 0$, such that (1.11)-(1.12) has a unique solution (x_h^m, y_h^m), $m \in [0, ..., M]$ and the following error bounds are satisfied*

$$\max_{m=0,\ldots,M} \left\|x^m - x_h^m\right\|_{H^1}^2 + \left\|y^m - y_h^m\right\|^2$$
$$+ \sum_{k=0}^{M-1} \Delta t \left(\left\|y_u^{k+1} - y_{hu}^{k+1}\right\|^2 + \left\|\frac{e^{k+1} - e^k}{\Delta t}\right\|^2\right) \leq C\left(h^2 + \Delta t^2\right)$$

for all $0 < h \leq h_0$, $0 < \Delta t \leq \Delta t_0$, provided $\Delta t \leq \mu h^5$. The constants depend only on the norms of the continuous solution.

In order to prove the error bounds between continuous and discrete solution, we set up the finite induction argument. To this purpose we introduce the following function

$$\rho^m := \left\|x^m - x_h^m\right\|^2 + \left\|y^m - y_h^m\right\|^2 + \left\|\left|x_u^m\right| - \left|x_{hu}^m\right|\right\|^2 + \sum_{k=0}^{m-1} \Delta t \left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2 \tag{4.7}$$

and formulate the next theorem.

Theorem 4.2 (Finite induction argument)**.** *There exist $h_0 > 0$, $\Delta t_0 > 0$, $\hat{c} > 0$, $\gamma > 0$ and $0 < \mu \leq 1$, such that for $0 \leq m \leq M$ holds*

$$\frac{1}{2}c_0 \leq |x_{hu}^m| \leq 2C_0, \quad |y_h^m| \leq 2C_0 \quad in\ [0, 2\pi], \quad \sum_{k=0}^{m} \Delta t \left\| y_{hu}^m \right\|_{L^\infty}^2 \leq 2C_0, \tag{4.8}$$

$$\rho^m \leq \hat{c} e^{\gamma m \Delta t} \left(h^2 + \Delta t^2 \right) \tag{4.9}$$

and for $k = 1, ..., m$ the following equations are satisfied

$$\begin{aligned} \int_0^{2\pi} I_h \left[\left(\frac{x_h^k - x_h^{k-1}}{\Delta t}, \phi_h \right) \right] |x_{hu}^{k-1}| - \int_0^{2\pi} \frac{\left(P_h^{k-1} y_{hu}^k, \phi_{hu} \right)}{\left| x_{hu}^k \right|} \\ - \frac{1}{2} \int_0^{2\pi} I_h \left[\left| y_h^k \right|^2 \right] \left(\tau_h^k, \phi_{hu} \right) + \int_0^{2\pi} \frac{\left(P_h^{k-1} y_{hu}^k, x_{hu}^k - x_{hu}^{k-1} \right)}{\left| x_{hu}^k \right|^3} \left(x_{hu}^k, \phi_{hu} \right) \\ + \lambda \int_0^{2\pi} \left(\tau_h^k, \phi_{hu} \right) = 0, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \int_0^{2\pi} I_h \left[\left(y_h^k, \psi_h \right) \right] \left| x_{hu}^k \right| - \int_0^{2\pi} I_h \left[\left(y_h^{k-1}, \psi_h \right) \right] \left| x_{hu}^{k-1} \right| \\ + \int_0^{2\pi} \frac{\left(P_h^{k-1} \left(x_{hu}^k - x_{hu}^{k-1} \right), \psi_{hu} \right)}{\left| x_{hu}^k \right|} = 0, \end{aligned} \tag{4.11}$$

for all $0 < h \leq h_0$, $0 < \Delta t \leq \Delta t_0$, provided that

$$\Delta t \leq \mu h^5. \tag{4.12}$$

The constants depend only on the norms of the continuous solution and final time T.

Proof. We shall prove the statement by induction on m.
Base. For $m = 0$ the claimed bounds on $|x_{hu}^m|$ and $|y_h^m|$ follow from (2.15). To prove the third inequality in (4.8), we adopt the result formulated in Lemma 4.3.

Lemma 4.3 ([22], Lemma 2.2)**.** *Suppose that $x_{h0} = I_h x_0$. Then for $0 < h \leq h_0$,*

$$\| y(\cdot, 0) - y_h(\cdot, 0) \| \leq Ch.$$

Using in addition inverse inequalities (2.11), (2.12) and interpolation estimate (2.6) we deduce

$$\begin{aligned} \Delta t \left\| y_{hu}^0 \right\|_{L^\infty}^2 &\leq C\Delta t \left\| \left(I_h y^0 \right)_u - y_{hu}^0 \right\|_{L^\infty}^2 + C\Delta t \left\| \left(I_h y^0 \right)_u - y_u^0 \right\|_{L^\infty}^2 + C\Delta t \left\| y_u^0 \right\|_{L^\infty}^2 \\ &\leq C\Delta t h^{-3} \left\| I_h y^0 - y_h^0 \right\|^2 + C\Delta t h^2 + C\Delta t \left\| y_u^0 \right\|_{L^\infty}^2 \\ &\leq C\Delta t h^{-3} \left\| y^0 - y_h^0 \right\|^2 + C\Delta t \left(1 + h + h^2 \right) \leq C\Delta t h^{-1} \leq C\mu h^4 \leq Ch^4 \leq 2C_0, \end{aligned}$$

if $h \leq h_0$ and (4.12) holds. Furthermore, recalling the initial conditions (2.13), (2.14) and using again (2.6) we infer

$$\rho^0 = \left\| x^0 - x_h^0 \right\|^2 + \left\| y^0 - y_h^0 \right\|^2 + \left\| \left| x_u^0 \right| - \left| x_{hu}^0 \right| \right\|^2 \leq Ch^2.$$

Assumption step. Let now $m \in [1, M-1]$ and suppose we have (x_h^m, y_h^m), such that the induction hypothesis is valid, i.e. (4.8) and (4.9) hold and equations (4.10)-(4.11) are satisfied. Further, Theorem 3.1 applies and yields the existence of $\left(x_h^{m+1}, y_h^{m+1}\right)$ satisfying (3.2) which solves the system

$$\begin{aligned}\int_0^{2\pi} I_h\left[\left(\frac{x_h^{m+1}-x_h^m}{\Delta t}, \phi_h\right)\right]|x_{hu}^m| - \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1}, \phi_{hu}\right)}{\left|x_{hu}^{m+1}\right|}\\ -\frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^{m+1}\right|^2\right]\left(\tau_h^{m+1}, \phi_{hu}\right) + \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1}-x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1}, \phi_{hu}\right)\\ +\lambda\int_0^{2\pi}\left(\tau_h^{m+1}, \phi_{hu}\right) = 0,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\int_0^{2\pi} I_h\left[\left(y_h^{m+1}, \psi_h\right)\right]\left|x_{hu}^{m+1}\right| - \int_0^{2\pi} I_h\left[\left(y_h^m, \psi_h\right)\right]|x_{hu}^m|\\ +\int_0^{2\pi}\frac{\left(P_h^m\left(x_{hu}^{m+1}-x_{hu}^m\right), \psi_{hu}\right)}{\left|x_{hu}^{m+1}\right|} = 0.\end{aligned} \tag{4.14}$$

Induction step. Let the assumption hold, then

$$\frac{1}{2}c_0 \le \left|x_{hu}^{m+1}\right| \le 2C_0, \quad \left|y_h^{m+1}\right| \le 2C_0 \quad \text{in } [0, 2\pi], \quad \sum_{k=0}^{m+1}\Delta t\left\|y_{hu}^k\right\|_{L^\infty}^2 \le 2C_0,$$
$$\rho^{m+1} \le \hat{c}e^{\gamma(m+1)\Delta t}\left(h^2+\Delta t^2\right). \tag{4.15}$$

To prove the induction step, we derive the estimates for the norms, which appear in the function ρ^{m+1}. In order to carry out the error analysis, we shall use the a priori bounds

$$\frac{1}{4}c_0 \le \left|x_{hu}^{m+1}\right| \le 4C_0, \quad \left|y_h^{m+1}\right| \le 4C_0 \quad \text{in } [0, 2\pi], \quad \sum_{k=0}^{m+1}\Delta t\left\|y_{hu}^k\right\|_{L^\infty}^2 \le 4C_0. \tag{4.16}$$

Here, the first two bounds follow directly from (3.2), whereas the third one can be derived using in addition (2.11), the induction hypothesis (4.8) and (4.12)

$$\begin{aligned}\sum_{k=0}^{m+1}\Delta t\,\|y_{hu}^m\|_{L^\infty}^2 =&\Delta t\left\|y_{hu}^{m+1}\right\|_{L^\infty}^2 + \sum_{k=0}^{m}\Delta t\,\|y_{hu}^m\|_{L^\infty}^2 \le C\Delta t h^{-2}\left\|y_h^{m+1}\right\|_{L^\infty}^2 + 2C_0\\ \le&16C_0^2C\Delta t h^{-2} + 2C_0 \le 16C_0^2C\mu h^3 + 2C_0 \le 16C_0^2Ch^3 + 2C_0 \le 4C_0,\end{aligned}$$

if h_0 is sufficiently small.
After the error analysis is finished, we can perform the induction step. The proof of the statement (4.15) is given in Theorem 5.2. Further, in Chapter 5.2 we improve the a priori bounds (4.16) and thereby complete the proof of the present theorem. □

Let us begin with deriving an error estimate for the scheme (4.13)-(4.14). In what follows, we shall assume that the bounds (4.8) hold. Additional assumptions on h, Δt will be formulated for each lemma separately. Below we present the results of the first lemma of the error analysis.

4.2 Position vector

From the error decomposition follows $\|\epsilon^m\| = \|I_h x^m - x^m\| \leq Ch^2$ by a standard interpolation estimate. Therefore it will be enough to prove the error bound for $e_h^m = I_h x^m - x_h^m$. Lemma 4.4 below and Lemma 4.5 in the next section derive an estimate for the discrete time derivative of the error in the position and curvature vector, respectively. These basic estimates will be combined in Lemma 4.6.

Lemma 4.4 (Position vector). *Suppose (4.16) holds. Then there exist $h_0 > 0$ and $\omega > 0$, such that for all $0 < h \leq h_0$, provided that $\Delta t \leq \omega h^3$ we have for $m = 0, ..., M-1$*

$$\begin{aligned}\frac{c_0}{4}\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2 &- \int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m y_u^{m+1} - \frac{1}{\left|x_{hu}^{m+1}\right|}P_h^m y_{hu}^{m+1}, \frac{e_u^{m+1}-e_u^m}{\Delta t}\right) \\ &- \frac{1}{2}\int_0^{2\pi}\left(\left|y^{m+1}\right|^2\tau^{m+1} - \left|y_h^{m+1}\right|^2\tau_h^{m+1}, \frac{e_u^{m+1}-e_u^m}{\Delta t}\right) \\ \leq C\Big(h^2 + \Delta t^2 &+ \||x_u^m| - |x_{hu}^m|\|^2 + \|\tau^m - \tau_h^m\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|_{H^1}^2 \\ &+ \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2\Big) + Ch^{-3}RT^{m+1},\end{aligned}$$

where

$$RT^{m+1} = \left(\sum_{k=0}^m \left\|\tau_h^{k+1} - \tau_h^k\right\|^2\right)^2. \tag{4.17}$$

The constants are independent of h and Δt.

Proof. To begin, we take the difference between the continuous equation (1.7) evaluated at $(m+1)\,\Delta t$ and ϕ_h and the discrete equation (4.13), what results in

$$\begin{aligned}&\int_0^{2\pi}\left(x_t^{m+1}, \phi_h\right)\left|x_u^{m+1}\right| - \int_0^{2\pi}\frac{1}{\left|x_u^{m+1}\right|}\left(P^{m+1}y_u^{m+1}, \phi_{hu}\right) \\ &\qquad - \frac{1}{2}\int_0^{2\pi}\left|y^{m+1}\right|^2\left(\tau^{m+1}, \phi_{hu}\right) + \lambda\int_0^{2\pi}\left(\tau^{m+1}, \phi_{hu}\right) \\ &- \Bigg(\int_0^{2\pi} I_h\left[\left(\frac{x_h^{m+1}-x_h^m}{\Delta t}, \phi_h\right)\right]|x_{hu}^m| - \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1}, \phi_{hu}\right)}{\left|x_{hu}^{m+1}\right|} \\ &\quad - \frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^{m+1}\right|^2\right]\left(\tau_h^{m+1}, \phi_{hu}\right) + \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1} - x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1}, \phi_{hu}\right) \\ &\qquad + \lambda\int_0^{2\pi}\left(\tau_h^{m+1}, \phi_{hu}\right)\Bigg) = 0\end{aligned}$$

for all $\phi_h \in X_h^n$.

Next, we add the discrete time derivative of the error to both sides of the above equation. After rearranging, one obtains

$$\begin{aligned}
&\int_0^{2\pi}\left(\frac{e_h^{m+1}-e_h^m}{\Delta t},\phi_h\right)|x_{hu}^m| - \int_0^{2\pi}\frac{(P^{m+1}y_u^{m+1},\phi_{hu})}{|x_u^{m+1}|} + \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1},\phi_{hu}\right)}{\left|x_{hu}^{m+1}\right|}\\
&\quad-\frac{1}{2}\int_0^{2\pi}\left|y^{m+1}\right|^2\left(\tau^{m+1},\phi_{hu}\right)+\frac{1}{2}\int_0^{2\pi}I_h\left[\left|y_h^{m+1}\right|^2\right]\left(\tau_h^{m+1},\phi_{hu}\right)\\
&=\int_0^{2\pi}\left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right],\phi_h\right)|x_{hu}^m| - \int_0^{2\pi}\left(x_t^{m+1},\phi_h\right)\left|x_u^{m+1}\right| \\
&\quad+\int_0^{2\pi}I_h\left[\left(\frac{x_h^{m+1}-x_h^m}{\Delta t},\phi_h\right)\right]|x_{hu}^m| - \int_0^{2\pi}\left(\frac{x_h^{m+1}-x_h^m}{\Delta t},\phi_h\right)|x_{hu}^m|\\
&\quad-\lambda\int_0^{2\pi}\left(\tau^{m+1}-\tau_h^{m+1},\phi_{hu}\right)+\int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1},x_{hu}^{m+1}-x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\phi_{hu}\right).
\end{aligned}\tag{4.18}$$

To combine the second and third terms on the left-hand side of (4.18), we require the projection matrix at the same time level. For simplicity, we take the previous time step. Thus, we add the desired term to the left-hand side of (4.18) and then correct the right-hand side with the corresponding difference. After some calculations, we arrive at

$$\begin{aligned}
&\int_0^{2\pi}\left(\frac{e_h^{m+1}-e_h^m}{\Delta t},\phi_h\right)|x_{hu}^m| - \int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m y_u^{m+1}-\frac{1}{\left|x_{hu}^{m+1}\right|}P_h^m y_{hu}^{m+1},\phi_{hu}\right)\\
&\quad-\frac{1}{2}\int_0^{2\pi}\left(\left|y^{m+1}\right|^2\tau^{m+1}-\left|y_h^{m+1}\right|^2\tau_h^{m+1},\phi_{hu}\right)\\
&=\int_0^{2\pi}\left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right],\phi_h\right)|x_{hu}^m| - \int_0^{2\pi}\left(x_t^{m+1},\phi_h\right)\left|x_u^{m+1}\right|\\
&\quad+\int_0^{2\pi}I_h\left[\left(\frac{x_h^{m+1}-x_h^m}{\Delta t},\phi_h\right)\right]|x_{hu}^m| - \int_0^{2\pi}\left(\frac{x_h^{m+1}-x_h^m}{\Delta t},\phi_h\right)|x_{hu}^m|\\
&\quad-\frac{1}{2}\int_0^{2\pi}I_h\left[\left|y_h^{m+1}\right|^2\right]\left(\tau_h^{m+1},\phi_{hu}\right)+\frac{1}{2}\int_0^{2\pi}\left|y_h^{m+1}\right|^2\left(\tau_h^{m+1},\phi_{hu}\right)\\
&\quad-\lambda\int_0^{2\pi}\left(\tau^{m+1},\phi_{hu}\right)+\lambda\int_0^{2\pi}\left(\tau_h^{m+1},\phi_{hu}\right)-\lambda\int_0^{2\pi}\left(y_h^{m+1},\phi_h\right)\left|x_{hu}^{m+1}\right|\\
&\quad+\lambda\int_0^{2\pi}\left(y_h^{m+1},\phi_h\right)\left|x_{hu}^{m+1}\right|-\int_0^{2\pi}\frac{1}{|x_u^{m+1}|}\left(\left(P^m-P^{m+1}\right)y_u^{m+1},\phi_{hu}\right)\\
&\quad+\int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1},x_{hu}^{m+1}-x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\phi_{hu}\right).
\end{aligned}\tag{4.19}$$

We note that the sixth term on the right-hand side of (4.19) is compensated with the one on the left-hand side. Moreover, for later use we have added to the right-hand of (4.19) the ninth and tenth terms, whose sum results in zero.

Let us rewrite several terms on the right-hand side of (4.19). In view of (1.8) one obtains

$$\lambda \int_0^{2\pi} \left(\tau^{m+1}, \phi_{hu}\right) = -\lambda \int_0^{2\pi} \left(y^{m+1}, \phi_h\right) \left|x_u^{m+1}\right|, \tag{4.20}$$

while Lemma 3.7 gives

$$\lambda \int_0^{2\pi} \left(\tau_h^{m+1}, \phi_{hu}\right) = -\lambda \int_0^{2\pi} I_h\left[\left(y_h^{m+1}, \phi_h\right)\right] \left|x_{hu}^{m+1}\right| - \lambda \left\langle R_h^{m+1}, \phi_h \right\rangle. \tag{4.21}$$

Furthermore, (2.8) implies

$$\begin{aligned}
&\int_0^{2\pi} I_h\left[\left(\frac{x_h^{m+1}-x_h^m}{\Delta t}, \phi_h\right)\right] |x_{hu}^m| - \int_0^{2\pi} \left(\frac{x_h^{m+1}-x_h^m}{\Delta t}, \phi_h\right) |x_{hu}^m| \\
&\quad - \frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^{m+1}\right|^2\right] \left(\tau_h^{m+1}, \phi_{hu}\right) + \frac{1}{2}\int_0^{2\pi} \left|y_h^{m+1}\right|^2 \left(\tau_h^{m+1}, \phi_{hu}\right) \\
&\quad + \lambda \int_0^{2\pi} \left(y_h^{m+1}, \phi_h\right) \left|x_{hu}^{m+1}\right| - \lambda \int_0^{2\pi} I_h\left[\left(y_h^{m+1}, \phi_h\right)\right] \left|x_{hu}^{m+1}\right| \\
&= \frac{1}{6}\sum_{j=1}^N h_j^2 \int_{I_j} \left(\left(\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}, \phi_{hu}\right) |x_{hu}^m| - \frac{1}{2}\left|y_{hu}^{m+1}\right|^2 \left(\tau_h^{m+1}, \phi_{hu}\right)\right) \\
&\quad - \frac{\lambda}{6}\sum_{j=1}^N h_j^2 \int_{I_j} (y_{hu}^{m+1}, \phi_{hu}) |x_{hu}^{m+1}|.
\end{aligned} \tag{4.22}$$

Inserting $\phi_h = \dfrac{e_h^{m+1}-e_h^m}{\Delta t}$ into (4.19) and recalling the error decomposition $e^m = \epsilon^m + e_h^m$, the equation (4.19) in view of the calculations (4.20)-(4.22) translates into

$$\begin{aligned}
&\int_0^{2\pi} \left|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right|^2 |x_{hu}^m| - \int_0^{2\pi} \left(\frac{P^m y_u^{m+1}}{|x_u^{m+1}|} - \frac{P_h^m y_{hu}^{m+1}}{\left|x_{hu}^{m+1}\right|}, \frac{e_u^{m+1}-e_u^m}{\Delta t}\right) \\
&\quad - \frac{1}{2}\int_0^{2\pi} \left(\left|y^{m+1}\right|^2 \tau^{m+1} - \left|y_h^{m+1}\right|^2 \tau_h^{m+1}, \frac{e_u^{m+1}-e_u^m}{\Delta t}\right) \\
&= \int_0^{2\pi} \left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right], \frac{e_h^{m+1}-e_h^m}{\Delta t}\right) |x_{hu}^m| - \int_0^{2\pi} \left(x_t^{m+1}, \frac{e_h^{m+1}-e_h^m}{\Delta t}\right) \left|x_u^{m+1}\right| \\
&\quad + \lambda \int_0^{2\pi} \left(y^{m+1}, \frac{e_h^{m+1}-e_h^m}{\Delta t}\right) \left|x_u^{m+1}\right| - \lambda \int_0^{2\pi} \left(y_h^{m+1}, \frac{e_h^{m+1}-e_h^m}{\Delta t}\right) |x_{hu}^{m+1}| \\
&\quad + \frac{1}{6}\sum_{j=1}^N h_j^2 \int_{I_j} \left(\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}, \frac{e_{hu}^{m+1}-e_{hu}^m}{\Delta t}\right) |x_{hu}^m| \\
&\quad - \frac{1}{6}\sum_{j=1}^N h_j^2 \int_{I_j} \frac{1}{2} |y_{hu}^{m+1}|^2 \left(\tau_h^{m+1}, \frac{e_{hu}^{m+1}-e_{hu}^m}{\Delta t}\right) \\
&\quad - \frac{1}{6}\sum_{j=1}^N h_j^2 \int_{I_j} \lambda \left(y_{hu}^{m+1}, \frac{e_{hu}^{m+1}-e_{hu}^m}{\Delta t}\right) \left|x_{hu}^{m+1}\right|
\end{aligned} \tag{4.23}$$

$$
\begin{aligned}
&-\int_0^{2\pi} \frac{1}{\left|x_u^{m+1}\right|}\left(\left(P^m-P^{m+1}\right) y_u^{m+1}, \frac{e_{hu}^{m+1}-e_{hu}^m}{\Delta t}\right) \\
&+\int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1}-x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1}, \frac{e_{hu}^{m+1}-e_{hu}^m}{\Delta t}\right)-\lambda\left\langle R_h^{m+1}, \frac{e_h^{m+1}-e_h^m}{\Delta t}\right\rangle \\
&-\int_0^{2\pi}\left(\frac{1}{\left|x_u^{m+1}\right|} P^m y_u^{m+1}-\frac{1}{\left|x_{hu}^{m+1}\right|} P_h^m y_{hu}^{m+1}, \frac{\epsilon_u^{m+1}-\epsilon_u^m}{\Delta t}\right) \\
&-\frac{1}{2} \int_0^{2\pi}\left(\left|y^{m+1}\right|^2 \tau^{m+1}-\left|y_h^{m+1}\right|^2 \tau_h^{m+1}, \frac{\epsilon_u^{m+1}-\epsilon_u^m}{\Delta t}\right) \\
=&\sum_{i=1}^{12} S_i .
\end{aligned}
$$

Let us next estimate the terms on the right-hand side of the equation (4.23). Firstly, we use the definition of ϵ^m to rewrite the sum of S_1 and S_2 as

$$
\begin{aligned}
S_1+S_2=&-\int_0^{2\pi}\left(\frac{\epsilon^{m+1}-\epsilon^m}{\Delta t}, \frac{e_h^{m+1}-e_h^m}{\Delta t}\right)\left|x_{hu}^m\right| \\
&+\int_0^{2\pi}\left(\frac{x^{m+1}-x^m}{\Delta t}-x_t^{m+1}, \frac{e_h^{m+1}-e_h^m}{\Delta t}\right)\left|x_{hu}^m\right| \\
&+\int_0^{2\pi}\left(x_t^{m+1}, \frac{e_h^{m+1}-e_h^m}{\Delta t}\right)\left(\left|x_{hu}^m\right|-\left|x_u^{m+1}\right|\right) \\
=&I+II+III .
\end{aligned}
$$

Thus, Cauchy-Schwarz and Young's inequalities along with (4.8) imply

$$
\begin{aligned}
I \leq & C\left(\int_0^{2\pi}\left|\frac{\epsilon^{m+1}-\epsilon^m}{\Delta t}\right|^2\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\left|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right|^2\right)^{\frac{1}{2}} \\
\leq & \varepsilon\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2+C_{\varepsilon} h^4\left\|\frac{x^{m+1}-x^m}{\Delta t}\right\|_{H^2}^2 \leq \varepsilon\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2+C_{\varepsilon} h^4,
\end{aligned}
$$

where the last estimates follow from (2.6) and the boundedness of the continuous solution. From the Taylor expansion we infer

$$
x^{m+1}-x^m=x_t^m \Delta t+O\left(\Delta t^2\right) \tag{4.24}
$$

and similarly as above obtain

$$
II \leq C\left(\int_0^{2\pi}\left|\frac{x^{m+1}-x^m}{\Delta t}-x_t^{m+1}\right|^2\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\left|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right|^2\right)^{\frac{1}{2}} \leq \varepsilon\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2+C_{\varepsilon} \Delta t^2 .
$$

Cauchy-Schwarz, Young's and reverse triangle inequalities along with the boundedness of the continuous solution provide the estimate for the third term

$$\begin{aligned} III \leq & C \int_0^{2\pi} \left| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right| \left(\left| |x_{hu}^m| - |x_u^m| \right| + \left| |x_u^{m+1}| - |x_u^m| \right| \right) \\ \leq & \varepsilon \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\|^2 + C_\varepsilon \left(\Delta t^2 + \| |x_u^m| - |x_{hu}^m| \|^2 \right). \end{aligned}$$

Rewriting the sum of S_3 and S_4 and using an upper bound on $|x_{hu}^{m+1}|$ from (4.16) we derive

$$\begin{aligned} S_3 + S_4 = & \lambda \int_0^{2\pi} \left(y^{m+1}, \frac{e_h^{m+1} - e_h^m}{\Delta t} \right) \left(|x_u^{m+1}| - |x_{hu}^{m+1}| \right) \\ & + \lambda \int_0^{2\pi} \left(y^{m+1} - y_h^{m+1}, \frac{e_h^{m+1} - e_h^m}{\Delta t} \right) |x_{hu}^{m+1}| \\ \leq & \varepsilon \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\|^2 + C_\varepsilon \left(\left\| |x_u^{m+1}| - |x_{hu}^{m+1}| \right\|^2 + \left\| y^{m+1} - y_h^{m+1} \right\|^2 \right). \end{aligned}$$

Let us consider the fifth term on the right-hand side of (4.23). Adding a zero term and recalling the definition of e_h^m yields

$$\begin{aligned} S_5 = & \frac{1}{6} \sum_{j=1}^N h_j^2 \int_{I_j} \left(\frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t} - I_h \left[\frac{x^{m+1} - x^m}{\Delta t} \right], \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) |x_{hu}^m| \\ & + \frac{1}{6} \sum_{j=1}^N h_j^2 \int_{I_j} \left(I_h \left[\frac{x^{m+1} - x^m}{\Delta t} \right], \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) |x_{hu}^m| \\ = & -\frac{1}{6} \sum_{j=1}^N h_j^2 \int_{I_j} \left| \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right|^2 |x_{hu}^m| + \frac{1}{6} \sum_{j=1}^N h_j^2 \int_{I_j} \left(I_h \left[\frac{x^{m+1} - x^m}{\Delta t} \right], \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) |x_{hu}^m| \\ = & S_{5,1} + S_{5,2}. \end{aligned}$$

We observe that $S_{5,1}$ is negative and therefore can be used for combination with positive terms of the same structure, which appear on the right-hand side. This combination can be then estimated by zero from above, provided such positive terms have a small factor $\delta > 0$ in front. To this end, we use Young's inequality in the next estimates. Thus, (4.4), (4.8) and Cauchy-Schwarz inequality imply

$$\begin{aligned} S_{5,2} = & \frac{1}{6} \sum_{j=1}^N h_j^2 \int_{I_j} \left(I_h \left[\frac{x^{m+1} - x^m}{\Delta t} \right], \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) |x_{hu}^m| \\ \leq & \delta \sum_{j=1}^N h_j^2 \int_{I_j} \left| \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right|^2 + C_\delta \sum_{j=1}^N h_j^2 \int_{I_j} |x_{hu}^m|^2 \\ \leq & \delta \sum_{j=1}^N h_j^2 \int_{I_j} \left| \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right|^2 + C_\delta h^2. \end{aligned}$$

In order to deal with S_6, we add a zero term and combine the resulting expressions as follows

$$\begin{aligned}
S_6 &= -\frac{1}{6}\sum_{j=1}^{N} h_j^2 \int_{I_j} \frac{1}{2}\left(\left|y_{hu}^{m+1}\right|^2 - \left|\left(I_h y^{m+1}\right)_u\right|^2 + \left|\left(I_h y^{m+1}\right)_u\right|^2\right)\left(\tau_h^{m+1}, \frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right)\\
&= -\frac{1}{6}\sum_{j=1}^{N} h_j^2 \int_{I_j} \frac{1}{2}\left(y_{hu}^{m+1} - \left(I_h y^{m+1}\right)_u, y_{hu}^{m+1} + \left(I_h y^{m+1}\right)_u\right)\left(\tau_h^{m+1}, \frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right)\\
&\quad -\frac{1}{6}\sum_{j=1}^{N} h_j^2 \int_{I_j} \frac{1}{2}\left|\left(I_h y^{m+1}\right)_u\right|^2\left(\tau_h^{m+1}, \frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right)\\
&= S_{6,1} + S_{6,2}.
\end{aligned}$$

The first term in S_6 can be estimated by taking out of the integral the L^∞-norm of $\left|y_{hu}^{m+1}\right|$ and $\left|\left(I_h y^{m+1}\right)_u\right|$. Using further the inverse inequality (2.11) together with (4.16) and the boundedness of the continuous solution we obtain

$$\begin{aligned}
S_{6,1} &\le C\sum_{j=1}^{N} h_j^2 \left(\|y_{hu}^{m+1}\|_{L^\infty} + \|\left(I_h y^{m+1}\right)_u\|_{L^\infty}\right)\int_{I_j}\left|y_{hu}^{m+1} - \left(I_h y^{m+1}\right)_u\right|\left|\frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right|\\
&\le C\sum_{j=1}^{N} h_j^2\, h^{-1}\left(\|y_{h}^{m+1}\|_{L^\infty} + \|I_h y^{m+1}\|_{L^\infty}\right)\int_{I_j}\left|y_{hu}^{m+1} - \left(I_h y^{m+1}\right)_u\right|\left|\frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right|\\
&\le C\sum_{j=1}^{N} h_j \int_{I_j}\left|y_{hu}^{m+1} - \left(I_h y^{m+1}\right)_u\right|\left|\frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right|\\
&\le \delta\sum_{j=1}^{N} h_j^2 \int_{I_j}\left|\frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right|^2 + C_\delta\left(h^2 + \|y_u^{m+1} - y_{hu}^{m+1}\|^2\right).
\end{aligned}$$

Here, an interpolation estimate together with Young's inequality completed the estimate. Next, in view of (4.3) we have

$$S_{6,2} \le \delta\sum_{j=1}^{N} h_j^2 \int_{I_j}\left|\frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right|^2 + C_\delta h^2.$$

Cauchy-Schwarz and triangle inequalities along with the bounds (4.2), (4.16) yield

$$\begin{aligned}
S_7 &= -\frac{1}{6}\sum_{j=1}^{N} h_j^2 \int_{I_j} \lambda\left(y_{hu}^{m+1}, \frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right)\left|x_{hu}^{m+1}\right|\\
&\le C\sum_{j=1}^{N} h_j^2 \int_{I_j}\left(\left|y_u^{m+1} - y_{hu}^{m+1}\right| + \left|y_u^{m+1}\right|\right)\left|\frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right|\\
&\le \delta\sum_{j=1}^{N} h_j^2 \int_{I_j}\left|\frac{e_{hu}^{m+1}-e_{hu}^{m}}{\Delta t}\right|^2 + C_\delta\left(h^2 + \|y_u^{m+1} - y_{hu}^{m+1}\|^2\right).
\end{aligned}$$

Combining now $S_{5,1}$ together with the estimates of the terms $S_{5,2}$, $S_{6,1}$ $S_{6,2}$ and S_7 we obtain after choosing δ small enough

$$-\frac{1}{6}\sum_{j=1}^{N} h_j^2 \int_{I_j} \left| \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right|^2 |x_{hu}^m| + 4\delta \sum_{j=1}^{N} h_j^2 \int_{I_j} \left| \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right|^2 \leq 0. \tag{4.25}$$

Exploiting the boundedness of the continuous solution, we integrate by parts the eighth term and arrive at

$$\begin{aligned} S_8 =& -\int_0^{2\pi} \frac{1}{|x_u^{m+1}|} \left(\left(P^m - P^{m+1}\right) y_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) \\ =& -\int_0^{2\pi} \frac{1}{|x_u^{m+1}|^3} \left(x_u^{m+1}, x_{uu}^{m+1}\right) \left(\left(P^m - P^{m+1}\right) y_u^{m+1}, \frac{e_h^{m+1} - e_h^m}{\Delta t} \right) \\ &+ \int_0^{2\pi} \frac{1}{|x_u^{m+1}|} \left(\left(P_u^m - P_u^{m+1}\right) y_u^{m+1}, \frac{e_h^{m+1} - e_h^m}{\Delta t} \right) \\ &+ \int_0^{2\pi} \frac{1}{|x_u^{m+1}|} \left(\left(P^m - P^{m+1}\right) y_{uu}^{m+1}, \frac{e_h^{m+1} - e_h^m}{\Delta t} \right) \leq \varepsilon \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\|^2 + C_\varepsilon \Delta t^2. \end{aligned}$$

Let us next examine S_9. To begin, we add a zero term and generate six differences

$$\begin{aligned} S_9 =& \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1} - x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) \\ =& \int_0^{2\pi} \left(P^m y_u^{m+1}, \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right) \left(x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) \left(\frac{1}{\left|x_{hu}^{m+1}\right|^3} - \frac{1}{|x_u^{m+1}|^3} \right) \\ &+ \int_0^{2\pi} \frac{\left(\left(P_h^m - P^m\right) y_u^{m+1}, \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) \\ &+ \int_0^{2\pi} \frac{\left(P_h^m y_u^{m+1}, \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1} - x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) \\ &+ \int_0^{2\pi} \frac{\left(P_h^m \left(y_{hu}^{m+1} - y_u^{m+1}\right), \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) \\ &+ \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1} - x_{hu}^m - \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) \\ &+ \int_0^{2\pi} \frac{\left(P^m y_u^{m+1}, \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right)}{|x_u^{m+1}|^3} \left(x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right) \\ =& S_{9,1} + S_{9,2} + S_{9,3} + S_{9,4} + S_{9,5} + S_{9,6}. \end{aligned}$$

Further, we treat these integrals separately, taking advantage of the boundedness of the continuous solution. Thus, Cauchy-Schwarz inequality, the inverse estimate (2.11) along with (4.16) imply

$$
\begin{aligned}
S_{9,1} =& \int_0^{2\pi} \left(P^m y_u^{m+1}, \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right) \left(x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right) \left(\frac{1}{\left|x_{hu}^{m+1}\right|^3} - \frac{1}{\left|x_u^{m+1}\right|^3}\right) \\
\leq& C\Delta t \int_0^{2\pi} \left|\frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right| \frac{\left|\left|x_u^{m+1}\right|^3 - \left|x_{hu}^{m+1}\right|^3\right|}{\left|x_{hu}^{m+1}\right|^3 \left|x_u^{m+1}\right|^3} \\
\leq& C\Delta t \left\|\frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right\| \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\| \leq C\Delta t h^{-1} \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\| \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\| \\
\leq& \left(\frac{\Delta t}{h}\right)^2 \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C\left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2.
\end{aligned}
$$

From the definition of the projection matrix (1.6) we deduce

$$
P_h^m - P^m = -\tau_h^m \otimes \tau_h^m + \tau^m \otimes \tau^m = -\left(\tau_h^m - \tau^m\right) \otimes \tau_h^m + \tau^m \otimes \left(\tau^m - \tau_h^m\right). \tag{4.26}
$$

Combining (4.24) and (4.26) and applying (4.16) we obtain for the second term

$$
\begin{aligned}
S_{9,2} =& \int_0^{2\pi} \frac{\left(\left(P_h^m - P^m\right) y_u^{m+1}, \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right) \\
\leq& C\Delta t \int_0^{2\pi} \left|\tau^m - \tau_h^m\right| \left|\frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right| \leq C\Delta t \left\|\tau^m - \tau_h^m\right\| \left\|\frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right\| \\
\leq& C\Delta t h^{-1} \left\|\tau^m - \tau_h^m\right\| \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\| \leq \left(\frac{\Delta t}{h}\right)^2 \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C\left\|\tau^m - \tau_h^m\right\|^2.
\end{aligned}
$$

Here, we have again used Cauchy-Schwarz inequality and the inverse estimate (2.11). Observing the following calculations

$$
\begin{aligned}
x_u^{m+1} - x_{hu}^{m+1} =& \tau^{m+1}\left|x_u^{m+1}\right| - \tau_h^{m+1}\left|x_{hu}^{m+1}\right| \\
=& \left(\tau^{m+1} - \tau_h^{m+1}\right)\left|x_u^{m+1}\right| + \tau_h^{m+1}\left(\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right)
\end{aligned} \tag{4.27}
$$

and taking into account (4.16), we deduce with the help of the inverse inequality (2.11)

$$
\begin{aligned}
S_{9,3} =& \int_0^{2\pi} \frac{\left(P_h^m y_u^{m+1}, \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1} - x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right) \\
\leq& C\Delta t \int_0^{2\pi} \left|x_u^{m+1} - x_{hu}^{m+1}\right| \left|\frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right| \\
\leq& C\Delta t h^{-1}\left(\left\|\tau^{m+1} - \tau_h^{m+1}\right\| + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|\right) \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\| \\
\leq& \left(\frac{\Delta t}{h}\right)^2 \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C\left(\left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2 + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2\right).
\end{aligned}
$$

We estimate the next term in a similar way as $S_{9,1}$

$$
\begin{aligned}
S_{9,4} =& \int_0^{2\pi} \frac{\left(P_h^m\left(y_{hu}^{m+1} - y_u^{m+1}\right), \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right) \\
\leq & C\Delta t \int_0^{2\pi} \left|y_u^{m+1} - y_{hu}^{m+1}\right| \left|\frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right| \leq C\Delta t \left\|y_u^{m+1} - y_{hu}^{m+1}\right\| \left\|\frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right\| \\
\leq & \left(\frac{\Delta t}{h}\right)^2 \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C\left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2.
\end{aligned}
$$

Recalling the definition of e_h^m and using afterwards the inverse inequality (2.11) as well as (4.16), we arrive at

$$
\begin{aligned}
S_{9,5} =& -\int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, e_{hu}^{m+1} - e_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right) \\
\leq & C\left\|y_{hu}^{m+1}\right\|_{L_\infty} \frac{\left\|e_{hu}^{m+1} - e_{hu}^m\right\|^2}{\Delta t} \leq C\Delta t h^{-3} \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2.
\end{aligned}
$$

It remains to estimate $S_{9,6}$. Inserting the continuous difference $x_u^{m+1} - x_u^m$ and integrating by parts the resulting second term as it is done in S_8 we deduce from (2.6) and (2.11)

$$
\begin{aligned}
S_{9,6} =& \int_0^{2\pi} \frac{\left(P^m y_u^{m+1}, \left(I_h\left[x^{m+1} - x^m\right]\right)_u - \left(x_u^{m+1} - x_u^m\right)\right)}{\left|x_u^{m+1}\right|^3} \left(x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right) \\
&+ \int_0^{2\pi} \frac{\left(P^m y_u^{m+1}, x_u^{m+1} - x_u^m\right)}{\left|x_u^{m+1}\right|^3} \left(x_u^{m+1}, \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t}\right) \\
\leq & C\Delta t \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\| \leq \varepsilon \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C_\varepsilon \Delta t^2.
\end{aligned}
$$

Next, we are going to handle S_{10} with the remainder term R_h^{m+1} given by (3.27). Factoring out the L^∞-norm and using further the inverse (2.12) and Young's inequalities we obtain

$$
\begin{aligned}
\left\langle R_h^{m+1}, \psi_h\right\rangle =& \sum_{k=0}^m \frac{1}{2} \int_0^{2\pi} \left(\tau_h^k, \psi_{hu}\right) \left|\tau_h^{k+1} - \tau_h^k\right|^2 \leq C\left\|\psi_{hu}\right\|_{L^\infty} \sum_{k=0}^m \left\|\tau_h^{k+1} - \tau_h^k\right\|^2 \\
\leq & Ch^{-\frac{3}{2}} \left\|\psi_h\right\| \sum_{k=0}^m \left\|\tau_h^{k+1} - \tau_h^k\right\|^2 \leq \varepsilon \left\|\psi_h\right\|^2 + C_\varepsilon h^{-3} RT^{m+1}.
\end{aligned}
$$

Here, for brevity we have used the notation (4.17). Setting now the test function $\psi_h = \dfrac{e_h^{m+1} - e_h^m}{\Delta t}$ into the above inequality we arrive at

$$
\left|S_{10}\right| \leq \varepsilon \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C_\varepsilon h^{-3} RT^{m+1}. \tag{4.28}
$$

Let us deal with S_{11}. To begin, we recall the definition of the Lagrange interpolation operator I_h and the definition (2.9) of Z_h and note

$$\int_0^{2\pi}\left(\frac{x_u^{m+1}-x_u^m}{\Delta t}, z_h\right) = \int_0^{2\pi}\left(\left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right]\right)_u, z_h\right), \quad \forall z_h \in Z_h.$$

From the above equality follows

$$\begin{aligned}
S_{11} =& \int_0^{2\pi}\left(\frac{P^m}{|x_u^{m+1}|}y_u^{m+1} - \frac{P_h^{m+1}}{|x_{hu}^{m+1}|}y_{hu}^{m+1}, \left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right]\right)_u - \frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\\
=& \int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m y_u^{m+1}, \left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right]\right)_u - \frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\\
=& \int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m y_u^{m+1}, \left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right]\right)_u - \frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\\
&- \int_0^{2\pi}\left(Q_h\left[\frac{1}{|x_u^{m+1}|}P^{m+1}y_u^{m+1}\right], \left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right]\right)_u - \frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\\
\leq& \left\|\frac{P^m}{|x_u^{m+1}|}y_u^{m+1} - Q_h\left[\frac{P^{m+1}}{|x_u^{m+1}|}y_u^{m+1}\right]\right\| \left\|\left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right]\right)_u - \frac{x_u^{m+1}-x_u^m}{\Delta t}\right\|\\
\leq& Ch\left\|\frac{1}{|x_u^{m+1}|}P^m y_u^{m+1}\right\|_{H^1} Ch \leq Ch^2,
\end{aligned} \tag{4.29}$$

where an interpolation estimate and (2.10) were used.

It remains to examine the last term on the right-hand side of (4.23). For this, we add zero terms and combine the resulting expressions in the following way

$$\begin{aligned}
S_{12} =& -\frac{1}{2}\int_0^{2\pi}\left(|y^{m+1}|^2\tau^{m+1} - \left|y_h^{m+1}\right|^2\tau_h^{m+1}, \frac{\epsilon_u^{m+1}-\epsilon_u^m}{\Delta t}\right)\\
=& -\frac{1}{2}\int_0^{2\pi}\left(\left|y^{m+1}\right|^2\left(\tau^{m+1}-\tau_h^{m+1}\right) + \left(\left|y^{m+1}\right|^2 - \left|y_h^{m+1}\right|^2\right)\tau_h^{m+1}, \frac{\epsilon_u^{m+1}-\epsilon_u^m}{\Delta t}\right)\\
\leq& C\int_0^{2\pi}\left(\left|y^{m+1}\right|^2\left|\tau^{m+1}-\tau_h^{m+1}\right| + \left|y^{m+1}-y_h^{m+1}\right|\left(\left|y^{m+1}\right| + \left|y_h^{m+1}\right|\right)\right)\left|\frac{\epsilon_u^{m+1}-\epsilon_u^m}{\Delta t}\right|\\
\leq& C\left(h^2 + \left\|\tau^{m+1}-\tau_h^{m+1}\right\|^2 + \left\|y^{m+1}-y_h^{m+1}\right\|^2\right).
\end{aligned}$$

Here, the last estimate follows from (2.6) and bounds (4.16).

Finally, in view of (4.8) we estimate from below the first term on the left-hand side of (4.23) and combine all the terms with the discrete time derivative of the error

$$\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2\left(\frac{c_0}{2} - 7\varepsilon - 4\Delta t^2 h^{-2} - C\Delta t h^{-3}\right) \geq \frac{c_0}{4}\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2.$$

The claim of the lemma follows by choosing $7\varepsilon = \frac{c_0}{12}$, $\omega = \frac{c_0}{12C}$ and provided h_0 is small enough. $\square$

4.3 Curvature vector

In this section, we shall be concerned with error bounds for the curvature vector.

Lemma 4.5 (Curvature vector). *Suppose that (4.16) holds. Then for* $\varepsilon > 0$ *and* $m = 0, ..., M-1$ *we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{1}{\Delta t}\left(\int_0^{2\pi} I_h\left[\left|I_h y^{m+1} - y_h^{m+1}\right|^2\right]\left|x_{hu}^{m+1}\right| - \int_0^{2\pi} I_h\left[\left|I_h y^m - y_h^m\right|^2\right]\left|x_{hu}^m\right|\right) \\
&+ \int_0^{2\pi}\left(\frac{1}{\left|x_u^{m+1}\right|}P^m\frac{x_u^{m+1}-x_u^m}{\Delta t} - \frac{1}{\left|x_{hu}^{m+1}\right|}P_h^m\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}, y_u^{m+1} - y_{hu}^{m+1}\right) \\
&- \frac{1}{2}\int_0^{2\pi}\left|y^{m+1} - y_h^{m+1}\right|^2 \frac{\left|x_{hu}^{m+1}\right| - \left|x_{hu}^m\right|}{\Delta t} \\
&+ \int_0^{2\pi}\left(\frac{\left|x_u^{m+1}\right| - \left|x_u^m\right|}{\Delta t}y^{m+1} - \frac{\left|x_{hu}^{m+1}\right| - \left|x_{hu}^m\right|}{\Delta t}y_h^{m+1}, y^{m+1} - y_h^{m+1}\right) \\
&\leq \varepsilon\left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C_\varepsilon\left(h^2 + \Delta t^2 + \left\||x_u^m| - |x_{hu}^m|\right\|^2 + \left\|\tau^m - \tau_h^m\right\|^2\right. \\
&\qquad\qquad\qquad\qquad \left. + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|_{H^1}^2\right).
\end{aligned}
\tag{4.30}
$$

Proof. Taking the difference between discrete time derivative of the equation (1.8) evaluated at $(m+1)\Delta t$ and ψ_h and the equation (1.12) divided by Δt we arrive at

$$
\begin{aligned}
&\frac{1}{\Delta t}\left(\int_0^{2\pi}\left(y^{m+1}, \psi_h\right)\left|x_u^{m+1}\right| + \int_0^{2\pi}\left(\tau^{m+1}, \psi_{hu}\right)\right. \\
&\qquad\qquad\qquad \left. - \int_0^{2\pi}\left(y^m, \psi_h\right)\left|x_u^m\right| - \int_0^{2\pi}\left(\tau^m, \psi_{hu}\right)\right) \\
&- \frac{1}{\Delta t}\left(\int_0^{2\pi} I_h\left[\left(y_h^{m+1}, \psi_h\right)\right]\left|x_{hu}^{m+1}\right| - \int_0^{2\pi} I_h\left[\left(y_h^m, \psi_h\right)\right]\left|x_{hu}^m\right|\right. \\
&\qquad\qquad\qquad \left. + \int_0^{2\pi}\frac{\left(P_h^m\psi_{hu}, x_{hu}^{m+1} - x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|}\right) = 0.
\end{aligned}
\tag{4.31}
$$

Comparing left-hand sides of (4.30), (4.31) we deduce that the term with the discrete projection matrix P_h^m, which appears in (4.31), requires generating the corresponding term at the continuous level. From (3.21) and the relation $P^m\tau^m = 0$, which follows from the definition of the projection matrix, we infer

$$
\begin{aligned}
\frac{P^m\left(x_u^{m+1} - x_u^m\right)}{\left|x_u^{m+1}\right|} &= P^m\tau^{m+1} - P^m\tau^m\frac{\left|x_u^m\right|}{\left|x_u^{m+1}\right|} = \tau^{m+1} - \tau^m\left(\tau^{m+1}, \tau^m\right) \\
&= \tau^{m+1} - \tau^m + \frac{1}{2}\tau^m\left|\tau^{m+1} - \tau^m\right|^2.
\end{aligned}
\tag{4.32}
$$

From (4.32) follows, the necessary term can be expressed with the help of the difference of the tangents on the left-hand side of (4.31).

One easily checks that (4.31) after some modifications and in view of (4.32) takes the form

$$
\begin{aligned}
&\int_0^{2\pi} I_h\left[\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right]-\frac{y_h^{m+1}-y_h^m}{\Delta t},\psi_h\right)\right]|x_{hu}^m|\\
&\quad+\int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m\frac{x_u^{m+1}-x_u^m}{\Delta t}-\frac{1}{|x_{hu}^{m+1}|}P_h^m\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t},\psi_{hu}\right)\\
&\quad+\int_0^{2\pi}\left(\frac{|x_u^{m+1}|-|x_u^m|}{\Delta t}y^{m+1}-\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}y_h^{m+1},\psi_h\right)\\
&=\int_0^{2\pi} I_h\left[\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right],\psi_h\right)\right]|x_{hu}^m|-\int_0^{2\pi}\left(\frac{y^{m+1}-y^m}{\Delta t},\psi_h\right)|x_u^m|\\
&\quad+\int_0^{2\pi} I_h\left[(y_h^{m+1},\psi_h)\right]\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}-\int_0^{2\pi}(y_h^{m+1},\psi_h)\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}\\
&\quad+\frac{1}{2}\int_0^{2\pi}(\tau^m,\psi_{hu})\frac{|\tau^{m+1}-\tau^m|^2}{\Delta t}.
\end{aligned}
\tag{4.33}
$$

From (2.8) we infer for the first term on the right-hand side of the above equation

$$
\begin{aligned}
\int_0^{2\pi} I_h\left[\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right],\psi_h\right)\right]|x_{hu}^m| &= \frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left(\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right]\right)_u,\psi_{hu}\right)|x_{hu}^m|\\
&\quad+\int_0^{2\pi}\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right],\psi_h\right)|x_{hu}^m|.
\end{aligned}
$$

Using (2.8) for the third and fourth terms on the right-hand side of (4.33) and the relation above, the equation (4.33) translates into

$$
\begin{aligned}
&\int_0^{2\pi} I_h\left[\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right]-\frac{y_h^{m+1}-y_h^m}{\Delta t},\psi_h\right)\right]|x_{hu}^m|\\
&\quad+\int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m\frac{x_u^{m+1}-x_u^m}{\Delta t}-\frac{1}{|x_{hu}^{m+1}|}P_h^m\left(\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right),\psi_{hu}\right)\\
&\quad+\int_0^{2\pi}\left(\frac{|x_u^{m+1}|-|x_u^m|}{\Delta t}y^{m+1}-\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}y_h^{m+1},\psi_h\right)\\
&=\int_0^{2\pi}\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right],\psi_h\right)|x_{hu}^m|-\int_0^{2\pi}\left(\frac{y^{m+1}-y^m}{\Delta t},\psi_h\right)|x_{hu}^m|\\
&\quad+\int_0^{2\pi}\left(\frac{y^{m+1}-y^m}{\Delta t},\psi_h\right)(|x_{hu}^m|-|x_u^m|)+\frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}(y_{hu}^{m+1},\psi_{hu})\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}\\
&\quad+\frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left(\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right]\right)_u,\psi_{hu}\right)|x_{hu}^m|+\frac{1}{2}\int_0^{2\pi}(\tau^m,\psi_{hu})\frac{|\tau^{m+1}-\tau^m|^2}{\Delta t}.
\end{aligned}
$$

Inserting $\psi_h = I_h y^{m+1} - y_h^{m+1}$ into the above equation we obtain

$$\begin{aligned}
&\int_0^{2\pi} I_h\left[\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right] - \frac{y_h^{m+1}-y_h^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right)\right]|x_{hu}^m| \\
&\quad + \int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m\frac{x_u^{m+1}-x_u^m}{\Delta t} - \frac{1}{|x_{hu}^{m+1}|}P_h^m\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right) \\
&\quad + \int_0^{2\pi}\left(\frac{|x_u^{m+1}|-|x_u^m|}{\Delta t}y^{m+1} - \frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}y_h^{m+1}, I_h y^{m+1} - y_h^{m+1}\right) \\
&= \int_0^{2\pi}\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right] - \frac{y^{m+1}-y^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right)|x_{hu}^m| \\
&\quad + \int_0^{2\pi}\left(\frac{y^{m+1}-y^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right)\left(|x_{hu}^m| - |x_u^m|\right) \\
&\quad + \frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left(\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right]\right)_u, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right)|x_{hu}^m| \\
&\quad + \frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left(y_{hu}^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right)\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t} \\
&\quad + \frac{1}{2}\int_0^{2\pi}\left(\tau^m, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}.
\end{aligned} \tag{4.34}$$

The elementary relation

$$\left(a^{m+1} - a^m, a^{m+1}\right) = \frac{1}{2}\left|a^{m+1}\right|^2 - \frac{1}{2}\left|a^m\right|^2 + \frac{1}{2}\left|a^{m+1} - a^m\right|^2 \tag{4.35}$$

allows us to rewrite the first term on the left-hand side of (4.34) as follows

$$\begin{aligned}
&\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[\left(\left(I_h y^{m+1} - y_h^{m+1}\right) - \left(I_h y^m - y_h^m\right), I_h y^{m+1} - y_h^{m+1}\right)\right]|x_{hu}^m| \\
&= \frac{1}{\Delta t}\left(\frac{1}{2}\int_0^{2\pi} I_h\left[\left|I_h y^{m+1} - y_h^{m+1}\right|^2\right]|x_{hu}^{m+1}| - \frac{1}{2}\int_0^{2\pi} I_h\left[\left|I_h y^m - y_h^m\right|^2\right]|x_{hu}^m|\right) \\
&\quad + \frac{1}{2}\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[\left|\left(I_h y^{m+1} - y_h^{m+1}\right) - \left(I_h y^m - y_h^m\right)\right|^2\right]|x_{hu}^m| \\
&\quad - \frac{1}{2}\int_0^{2\pi} I_h\left[\left|I_h y^{m+1} - y_h^{m+1}\right|^2\right]\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}.
\end{aligned}$$

Furthermore, (2.8) implies

$$\begin{aligned}
&\frac{1}{2}\int_0^{2\pi} I_h\left[\left|I_h y^{m+1} - y_h^{m+1}\right|^2\right]\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t} - \frac{1}{2}\int_0^{2\pi}\left|I_h y^{m+1} - y_h^{m+1}\right|^2\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t} \\
&= \frac{1}{2}\frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left|\left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right|^2\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}.
\end{aligned}$$

A scalar product in the second term on the left-hand side of (4.34) after inserting a zero term becomes $\left(\cdot, (I_h y^{m+1})_u - y_{hu}^{m+1}\right) = \left(\cdot, y_u^{m+1} - y_{hu}^{m+1}\right) + (\cdot, (I_h y^{m+1})_u - y_u^{m+1})$. In view of the above calculations, the equation (4.34) takes the form

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\frac{1}{2}\int_0^{2\pi} I_h\left[\left|I_h y^{m+1} - y_h^{m+1}\right|^2\right]|x_{hu}^{m+1}| - \frac{1}{2}\int_0^{2\pi} I_h\left[|I_h y^m - y_h^m|^2\right]|x_{hu}^m|\right)\\
&\quad + \int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m\frac{x_u^{m+1}-x_u^m}{\Delta t} - \frac{1}{|x_{hu}^{m+1}|}P_h^m\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}, y_u^{m+1} - y_{hu}^{m+1}\right)\\
&\quad + \int_0^{2\pi}\left(\frac{|x_u^{m+1}|-|x_u^m|}{\Delta t}y^{m+1} - \frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}y_h^{m+1}, I_h y^{m+1} - y_h^{m+1}\right)\\
&\quad - \frac{1}{2}\int_0^{2\pi}\left|I_h y^{m+1} - y_h^{m+1}\right|^2\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}\\
&\quad + \frac{1}{2}\frac{1}{\Delta t}\int_0^{2\pi} I_h\left[\left|\left(I_h y^{m+1} - y_h^{m+1}\right) - (I_h y^m - y_h^m)\right|^2\right]|x_{hu}^m|\\
&= \int_0^{2\pi}\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right] - \frac{y^{m+1}-y^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right)|x_{hu}^m|\\
&\quad + \int_0^{2\pi}\left(\frac{y^{m+1}-y^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right)(|x_{hu}^m| - |x_u^m|)\\
&\quad + \frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left(\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right]\right)_u, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right)|x_{hu}^m|\\
&\quad + \frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left(y_{hu}^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right)\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}\\
&\quad + \frac{1}{6}\sum_{j=1}^N h_j^2\int_{I_j}\frac{1}{2}\left|\left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right|^2\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}\\
&\quad + \frac{1}{2}\int_0^{2\pi}\left(\tau^m, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right)\frac{|\tau^{m+1}-\tau^m|^2}{\Delta t}\\
&\quad - \int_0^{2\pi}\left(\frac{1}{|x_u^{m+1}|}P^m\frac{x_u^{m+1}-x_u^m}{\Delta t} - \frac{1}{|x_{hu}^{m+1}|}P_h^m\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}, \left(I_h y^{m+1}\right)_u - y_u^{m+1}\right)\\
&= \sum_{i=1}^6 S_i.
\end{aligned} \tag{4.36}$$

Further, to estimate the terms on the right-hand side of (4.36) we use the boundedness of the continuous solution. From (3.1) and an interpolation estimate we obtain for the first term

$$\begin{aligned}
S_1 =& \int_0^{2\pi}\left(I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right] - \frac{y^{m+1}-y^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right)|x_{hu}^m|\\
\le& C\left\|I_h\left[\frac{y^{m+1}-y^m}{\Delta t}\right] - \frac{y^{m+1}-y^m}{\Delta t}\right\|\left\|I_h y^{m+1} - y_h^{m+1}\right\| \le C\left(h^4 + \left\|y^{m+1} - y_h^{m+1}\right\|^2\right).
\end{aligned}$$

Analogously, we derive

$$\begin{aligned}
S_2 =& \int_0^{2\pi} \left(\frac{y^{m+1} - y^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1} \right) (|x_{hu}^m| - |x_u^m|) \\
\leq& C \left\| I_h y^{m+1} - y_h^{m+1} \right\| \| |x_u^m| - |x_{hu}^m| \| \leq C \left(h^4 + \left\| y^{m+1} - y_h^{m+1} \right\|^2 + \| |x_u^m| - |x_{hu}^m| \|^2 \right).
\end{aligned}$$

With the help of the inverse inequality (2.11) and an interpolation estimate we deduce

$$\begin{aligned}
S_3 =& \frac{1}{6} \sum_{j=1}^{N} h_j^2 \int_{I_j} \left(\left(I_h \left[\frac{y^{m+1} - y^m}{\Delta t} \right] \right)_u, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) |x_{hu}^m| \\
\leq& C h^2 \left\| \left(I_h \left[\frac{y^{m+1} - y^m}{\Delta t} \right] \right)_u \right\| \left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| \\
\leq& C h \left\| I_h y^{m+1} - y_h^{m+1} \right\| \leq C \left(h^2 + \left\| y^{m+1} - y_h^{m+1} \right\|^2 \right).
\end{aligned}$$

To combine S_4 and S_5, we first observe

$$\begin{aligned}
&\left(y_{hu}^{m+1}, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) + \frac{1}{2} \left| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right|^2 \\
&= \frac{1}{2} \left(\left(I_h y^{m+1} \right)_u + y_{hu}^{m+1}, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right).
\end{aligned}$$

Using (2.11) and (4.16), we arrive at

$$\begin{aligned}
S_4 + S_5 =& \frac{1}{12} \sum_{j=1}^{N} h_j^2 \int_{I_j} \left(\left(I_h y^{m+1} \right)_u + y_{hu}^{m+1}, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} \\
\leq& C h^2 \left(\left\| \left(I_h y^{m+1} \right)_u \right\|_{L_\infty} + \left\| y_{hu}^{m+1} \right\|_{L_\infty} \right) \left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| \left\| \frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t} \right\| \\
\leq& C h \left(\left\| I_h y^{m+1} \right\|_{L_\infty} + \left\| y_h^{m+1} \right\|_{L_\infty} \right) \left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| \left\| \frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t} \right\| \\
\leq& C h \left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| \left(\left\| \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right\| + \left\| \left(I_h \left[\frac{x_u^{m+1} - x_u^m}{\Delta t} \right] \right)_u \right\| \right) \\
\leq& C h \left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| \left(h^{-1} \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\| + C \right) \\
\leq& \frac{2\varepsilon}{3} \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\|^2 + C_\varepsilon \left(h^2 + \left\| y_u^{m+1} - y_{hu}^{m+1} \right\|^2 \right).
\end{aligned}$$

The sixth term in view of (2.6) can be estimated as

$$\begin{aligned}
S_6 =& -\frac{1}{2} \int_0^{2\pi} \left(\tau^m, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) \frac{|\tau^{m+1} - \tau^m|^2}{\Delta t} \\
\leq& C \Delta t \int_0^{2\pi} \left| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right| \leq C \left(h^2 + \Delta t^2 + \left\| y_u^{m+1} - y_{hu}^{m+1} \right\|^2 \right).
\end{aligned}$$

Approaching the next term as it done in (4.29) in the proof of Lemma 4.4, we obtain

$$S_7 = \int_0^{2\pi} \left(\frac{1}{|x_u^{m+1}|} P^m \frac{x_u^{m+1} - x_u^m}{\Delta t} - \frac{1}{|x_{hu}^{m+1}|} P_h^m \frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}, \left(I_h y^{m+1}\right)_u - y_u^{m+1} \right) \leq Ch^2.$$

Let us rewrite the third and fourth terms on the left-hand side of (4.36) in the way, they appear in the formulation (4.30) of the lemma. A straightforward calculation shows

$$\begin{aligned}
& -\frac{1}{2} \left| I_h y^{m+1} - y_h^{m+1} \right|^2 \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} \\
& + \left(\frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} y^{m+1} - \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} y_h^{m+1}, I_h y^{m+1} - y_h^{m+1} \right) \\
= & -\frac{1}{2} \left| I_h y^{m+1} - y^{m+1} \right|^2 \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} - \frac{1}{2} \left| y^{m+1} - y_h^{m+1} \right|^2 \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} \\
& - \left(I_h y^{m+1} - y^{m+1}, y^{m+1} - y_h^{m+1} \right) \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} \\
& + \left(\frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} y^{m+1} - \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} y_h^{m+1}, y^{m+1} - y_h^{m+1} \right) \\
& - \left(\frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} y^{m+1} - \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} y_h^{m+1}, y^{m+1} - I_h y^{m+1} \right).
\end{aligned}$$

After simplifying one obtains

$$\begin{aligned}
& -\frac{1}{2} \left| I_h y^{m+1} - y^{m+1} \right|^2 \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} - \frac{1}{2} \left| y^{m+1} - y_h^{m+1} \right|^2 \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} \\
& + \left(\frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} y^{m+1} - \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} y_h^{m+1}, y^{m+1} - y_h^{m+1} \right) \\
& + \left(\frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} - \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} \right) \left(y^{m+1}, I_h y^{m+1} - y^{m+1} \right) \\
= & \, I + II + III + IV.
\end{aligned} \tag{4.37}$$

We observe that II and III in (4.37) correspond to the third and fourth terms on the left-hand side of (4.30). Therefore, we move I and IV to the right-hand side and estimate them. Using the following relation for arbitrary vectors $a, b \in \mathbb{R}^n$

$$|a| - |b| = \frac{(|a| - |b|)(|a| + |b|)}{|a| + |b|} = \frac{(a - b, a + b)}{|a| + |b|}$$

the fourth term IV can be rewritten as

$$\begin{aligned}
IV = & \left(\frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} - \frac{|x_{hu}^{m+1}| - |x_{hu}^m|}{\Delta t} \right) \left(y^{m+1}, I_h y^{m+1} - y^{m+1} \right) \\
= & - \left(\left(\frac{x_u^{m+1} - x_u^m}{\Delta t} - \frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}, \frac{x_u^{m+1} + x_u^m}{|x_u^{m+1}| + |x_u^m|} \right) \right. \\
& \left. + \left(\frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}, \frac{x_u^{m+1} + x_u^m}{|x_u^{m+1}| + |x_u^m|} - \frac{x_{hu}^{m+1} + x_{hu}^m}{|x_{hu}^{m+1}| + |x_{hu}^m|} \right) \right) \left(y^{m+1}, y^{m+1} - I_h y^{m+1} \right).
\end{aligned}$$

One can notice that the term $\dfrac{x_u^{m+1}+x_u^m}{|x_u^{m+1}|+|x_u^m|}-\dfrac{x_{hu}^{m+1}+x_{hu}^m}{|x_{hu}^{m+1}|+|x_{hu}^m|}$ behaves like a difference of the tangents. Let us show this explicitly. Finding the common denominator and splitting then the fractions results in

$$
\begin{aligned}
&\frac{x_u^{m+1}+x_u^m}{|x_u^{m+1}|+|x_u^m|}-\frac{x_{hu}^{m+1}+x_{hu}^m}{|x_{hu}^{m+1}|+|x_{hu}^m|}=\frac{\tau^{m+1}\left|x_u^{m+1}\right|+\tau^m|x_u^m|}{|x_u^{m+1}|+|x_u^m|}-\frac{\tau_h^{m+1}\left|x_{hu}^{m+1}\right|+\tau_h^m|x_{hu}^m|}{|x_{hu}^{m+1}|+|x_{hu}^m|}\\
&=\frac{\left(\tau^{m+1}\left|x_u^{m+1}\right|+\tau^m|x_u^m|\right)\left(|x_{hu}^{m+1}|+|x_{hu}^m|\right)-\left(\tau_h^{m+1}\left|x_{hu}^{m+1}\right|+\tau_h^m|x_{hu}^m|\right)\left(|x_u^{m+1}|+|x_u^m|\right)}{\left(|x_u^{m+1}|+|x_u^m|\right)\left(\left|x_{hu}^{m+1}\right|+|x_{hu}^m|\right)}\\
&=\frac{\left(\tau^{m+1}-\tau_h^{m+1}\right)|x_u^{m+1}|\left|x_{hu}^{m+1}\right|+\left(\tau^m-\tau_h^m\right)|x_u^m||x_{hu}^m|}{\left(|x_u^{m+1}|+|x_u^m|\right)\left(\left|x_{hu}^{m+1}\right|+|x_{hu}^m|\right)}\\
&\quad+\frac{\tau^{m+1}|x_u^{m+1}|\,|x_{hu}^m|+\tau^m|x_u^m|\left|x_{hu}^{m+1}\right|-\tau_h^{m+1}|x_u^m|\left|x_{hu}^{m+1}\right|-\tau_h^m\left|x_u^{m+1}\right||x_{hu}^m|}{\left(|x_u^{m+1}|+|x_u^m|\right)\left(\left|x_{hu}^{m+1}\right|+|x_{hu}^m|\right)}.
\end{aligned}
$$

Furthermore, the numerator of the second fraction transforms into

$$
\begin{aligned}
&\tau^{m+1}\left|x_u^{m+1}\right||x_{hu}^m|-\tau_h^m\left|x_u^{m+1}\right||x_{hu}^m|+\tau^m|x_u^m|\left|x_{hu}^{m+1}\right|-\tau_h^{m+1}\left|x_u^m\right|\left|x_{hu}^{m+1}\right|\\
&=\left(\tau^{m+1}-\tau^m\right)\left(\left|x_u^{m+1}\right||x_{hu}^m|-|x_u^m|\left|x_{hu}^{m+1}\right|\right)+\left(\tau^m-\tau_h^m\right)\left|x_u^{m+1}\right||x_{hu}^m|\\
&\quad+\left(\tau^{m+1}-\tau_h^{m+1}\right)|x_u^m|\left|x_{hu}^{m+1}\right|.
\end{aligned}
$$

Finally, using the above calculations we derive the last estimate. Recalling the error decomposition and exploiting (2.6), (2.11), (4.8) and (4.16) we deduce

$$
\begin{aligned}
-\int_0^{2\pi} I+IV =&\int_0^{2\pi}\frac{1}{2}\left|I_hy^{m+1}-y^{m+1}\right|^2\left(\frac{|x_{hu}^{m+1}|-|x_{hu}^m|}{\Delta t}\right)\\
&+\left(\left(\frac{x_u^{m+1}-x_u^m}{\Delta t}-\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t},\frac{x_u^{m+1}+x_u^m}{|x_u^{m+1}|+|x_u^m|}\right)\right.\\
&+\left.\left(\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t},\frac{x_u^{m+1}+x_u^m}{|x_u^{m+1}|+|x_u^m|}-\frac{x_{hu}^{m+1}+x_{hu}^m}{|x_{hu}^{m+1}|+|x_{hu}^m|}\right)\right)\left(y^{m+1},y^{m+1}-I_hy^{m+1}\right)\\
\le& C\int_0^{2\pi}\left|I_hy^{m+1}-y^{m+1}\right|^2\left|\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right|+\left|\frac{e_u^{m+1}-e_u^m}{\Delta t}\right|\left|I_hy^{m+1}-y^{m+1}\right|\\
&+\left|\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right|\left(\Delta t+|\tau^m-\tau_h^m|+\left|\tau^{m+1}-\tau_h^{m+1}\right|\right)\left|I_hy^{m+1}-y^{m+1}\right|\\
\le& C\left\|I_hy^{m+1}-y^{m+1}\right\|\left\|\frac{e_u^{m+1}-e_u^m}{\Delta t}\right\|+C\left\|I_hy^{m+1}-y^{m+1}\right\|_{L^\infty}\left\|\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right\|\\
&\left(\left\|I_hy^{m+1}-y^{m+1}\right\|+\Delta t+\|\tau^m-\tau_h^m\|+\left\|\tau^{m+1}-\tau_h^{m+1}\right\|\right)\\
\le& Ch^2\left\|\frac{e_u^{m+1}-e_u^m}{\Delta t}\right\|+Ch^2\left\|\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right\|\left(h^2+\Delta t+\|\tau^m-\tau_h^m\|+\left\|\tau^{m+1}-\tau_h^{m+1}\right\|\right)\\
\le& Ch^2\left\|\frac{\epsilon_u^{m+1}-\epsilon_u^m}{\Delta t}\right\|+Ch^2\left\|\frac{e_{hu}^{m+1}-e_{hu}^m}{\Delta t}\right\|\left(1+h^2+\Delta t+\|\tau^m-\tau_h^m\|+\left\|\tau^{m+1}-\tau_h^{m+1}\right\|\right)\\
&+Ch^2\left\|\left(I_h\left[\frac{x^{m+1}-x^m}{\Delta t}\right]\right)_u\right\|\left(h^2+\Delta t+\|\tau^m-\tau_h^m\|+\left\|\tau^{m+1}-\tau_h^{m+1}\right\|\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq Ch \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\| \left(1 + h^2 + \Delta t + \|\tau^m - \tau_h^m\| + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|\right) \\
&\quad + Ch^3 + Ch^2 \left(h^2 + \Delta t + \|\tau^m - \tau_h^m\| + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|\right) \\
&\leq \frac{\varepsilon}{3} \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\|^2 + C_\varepsilon \left(h^2 + \Delta t^2 + \|\tau^m - \tau_h^m\|^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2\right).
\end{aligned}
$$

Note that the last term on the left-hand side of (4.36) is non-negative and therefore can be estimated by zero from below. Thus, we obtain the claim of the lemma.

□

4.4 Combined result

In the following Lemma 4.6 we combine the estimates from Lemma 4.4 and Lemma 4.5. At the end of the error analysis this result is discretely integrated with respect to time, i.e. multiplied by Δt and summed over m. Finally, the discrete Gronwall's argument completes the proof of error bounds.

Lemma 4.6 (Combined result)**.** *Suppose (4.16) holds. Then there exists $\omega > 0$, such that for $\Delta t \leq \omega h^2$ and $m = 0, ..., M-1$ holds*

$$
\begin{aligned}
&\frac{c_0}{16} \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\|^2 + \frac{\zeta^{m+1} - \zeta^m}{\Delta t} \leq C \left(h^2 + \Delta t^2 + \|y^m - y_h^m\|_{H^1}^2 + \||x_u^m| - |x_{hu}^m|\|^2\right. \\
&\left. + \|\tau^m - \tau_h^m\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|_{H^1}^2 + \left\||x_u^{m+1}| - |x_{hu}^{m+1}|\right\|^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2\right) \\
&+ Ch^{-3} RT^{m+1},
\end{aligned}
\tag{4.38}
$$

where

$$
\begin{aligned}
\zeta^m = &\frac{1}{2} \int_0^{2\pi} I_h \left[|I_h y^m - y_h^m|^2\right] |x_{hu}^m| - \frac{1}{4} \int_0^{2\pi} |y^m|^2 |\tau^m - \tau_h^m|^2 |x_{hu}^m| \\
&+ \int_0^{2\pi} \left(\left(\frac{|x_{hu}^m| - |x_u^m|}{|x_u^m|}\right)(\tau_h^m - \tau^m) + \frac{1}{2} \frac{|x_{hu}^m|}{|x_u^m|} |\tau_h^m - \tau^m|^2 \tau^m, y_u^m\right),
\end{aligned}
\tag{4.39}
$$

RT^{m+1} is given by (4.17). The constants depend only on the norms of the continuous solution.

Proof. It follows from Lemma 4.4 and Lemma 4.5 with $\varepsilon = \dfrac{c_0}{8}$ that

$$
\begin{aligned}
&\frac{c_0}{8} \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\|^2 + \int_0^{2\pi} A^m + B^m \\
&\quad + \frac{1}{2} \frac{1}{\Delta t} \left(\int_0^{2\pi} I_h \left[\left|I_h y^{m+1} - y_h^{m+1}\right|^2\right] \left|x_{hu}^{m+1}\right| - \int_0^{2\pi} I_h \left[|I_h y^m - y_h^m|^2\right] |x_{hu}^m| \right) \\
&\leq Ch^{-3} RT^{m+1} + C \left(h^2 + (\Delta t)^2 + \||x_u^m| - |x_{hu}^m|\|^2 + \|\tau^m - \tau_h^m\|^2\right. \\
&\qquad \left. + \left\||x_u^{m+1}| - |x_{hu}^{m+1}|\right\|^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|_{H^1}^2\right),
\end{aligned}
\tag{4.40}
$$

where we have abbreviated

$$A^m = -\frac{1}{2}\left(\left|y^{m+1}\right|^2 \tau^{m+1} - \left|y_h^{m+1}\right|^2 \tau_h^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t} - \frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}\right)$$
$$- \frac{1}{2}\left|y^{m+1} - y_h^{m+1}\right|^2 \frac{\left|x_{hu}^{m+1}\right| - \left|x_{hu}^m\right|}{\Delta t}$$
$$+ \left(\frac{\left|x_u^{m+1}\right| - \left|x_u^m\right|}{\Delta t} y^{m+1} - \frac{\left|x_{hu}^{m+1}\right| - \left|x_{hu}^m\right|}{\Delta t} y_h^{m+1}, y^{m+1} - y_h^{m+1}\right),$$

$$B^m = \left(\frac{1}{\left|x_u^{m+1}\right|} P^m \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}\right) - \frac{1}{\left|x_{hu}^{m+1}\right|} P_h^m \left(\frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}\right), y_u^{m+1} - y_{hu}^{m+1}\right)$$
$$- \left(\frac{1}{\left|x_u^{m+1}\right|} P^m y_u^{m+1} - \frac{1}{\left|x_{hu}^{m+1}\right|} P_h^m y_{hu}^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t} - \frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}\right).$$

Our aim now is to organize terms in A^m and B^m in such a way that each of them is cubic in an appropriate difference or combination of the terms can be written as a discrete time derivative. In the first case, such terms will be estimated from below, whereas in the second case they will be taken into the difference $\zeta^{m+1} - \zeta^m$.

Derivation of A^m

A straightforward calculation shows

$$A^m = -\frac{1}{2}\left|y^{m+1}\right|^2 \frac{\left|x_{hu}^{m+1}\right| - \left|x_{hu}^m\right|}{\Delta t} + \frac{1}{2}\left|y^{m+1}\right|^2 \left(\tau^{m+1}, \frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}\right)$$
$$- \frac{1}{2}\left|y^{m+1}\right|^2 \left(\tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) + \left|y^{m+1}\right|^2 \frac{\left|x_u^{m+1}\right| - \left|x_u^m\right|}{\Delta t}$$
$$+ \frac{1}{2}\left|y_h^{m+1}\right|^2 \left(\tau_h^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) - \left(y^{m+1}, y_h^{m+1}\right) \frac{\left|x_u^{m+1}\right| - \left|x_u^m\right|}{\Delta t}$$
$$+ \frac{1}{2}\left|y_h^{m+1}\right|^2 \frac{\left|x_{hu}^{m+1}\right| - \left|x_{hu}^m\right|}{\Delta t} - \frac{1}{2}\left|y_h^{m+1}\right|^2 \left(\tau_h^{m+1}, \frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}\right).$$

Further, with the help of (3.21) we derive

$$A^m = -\frac{1}{2}\left|y^{m+1}\right|^2 \left(\frac{\left|x_{hu}^{m+1}\right| - \left|x_{hu}^m\right|}{\Delta t} - \frac{\left(x_{hu}^{m+1}, \tau^{m+1}\right) - \left(x_{hu}^m, \tau^m\right)}{\Delta t}\right)$$
$$- \frac{1}{2}\left|y^{m+1}\right|^2 \left(x_{hu}^m, \frac{\tau^{m+1} - \tau^m}{\Delta t}\right) - \frac{1}{2}\left|y^{m+1}\right|^2 \left(\tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right)$$
$$+ \left|y^{m+1}\right|^2 \frac{\left|x_u^{m+1}\right| - \left|x_u^m\right|}{\Delta t} + \frac{1}{2}\left|y_h^{m+1}\right|^2 \left(\tau_h^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right)$$
$$- \left(y^{m+1}, y_h^{m+1}\right) \frac{\left|x_u^{m+1}\right| - \left|x_u^m\right|}{\Delta t} - \frac{1}{4}\left|y_h^{m+1}\right|^2 \frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t}\left|x_{hu}^m\right|.$$

Next, we combine the terms in A^m by adding and subtracting several expressions to obtain

$$
\begin{aligned}
A^m =& \frac{1}{2}\left|y^{m+1}\right|^2\left(-\frac{\left|x_{hu}^{m+1}\right|-\left|x_{hu}^m\right|}{\Delta t}+\frac{\left(x_{hu}^{m+1},\tau^{m+1}\right)-\left(x_{hu}^m,\tau^m\right)}{\Delta t}\right)\\
&+\left\{\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)-\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\right\}\\
&+\left\{\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)-\left(y^{m+1},y_h^{m+1}\right)\left(\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\right.\\
&\quad\left.+\frac{1}{2}\left|y_h^{m+1}\right|^2\left(\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\right\}-\frac{1}{4}\left|y_h^{m+1}\right|^2\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t}\left|x_{hu}^m\right|\\
&+\left\{\left|y^{m+1}\right|^2\left(\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)-\left|y^{m+1}\right|^2\left(\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\right.\\
&\quad\left.-\left(y^{m+1},y_h^{m+1}\right)\left(\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)+\left(y^{m+1},y_h^{m+1}\right)\left(\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\right\}\\
&+\left\{-\left|y^{m+1}\right|^2\left(\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)+\left|y^{m+1}\right|^2\frac{\left|x_u^{m+1}\right|-\left|x_u^m\right|}{\Delta t}\right.\\
&\quad\left.-\left(y^{m+1},y_h^{m+1}\right)\frac{\left|x_u^{m+1}\right|-\left|x_u^m\right|}{\Delta t}+\left(y^{m+1},y_h^{m+1}\right)\left(\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\right\}\\
&+\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau_h^m,\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left|x_{hu}^m\right|-\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1},\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left|x_{hu}^m\right|\\
=& I+\ldots+VIII.
\end{aligned}
$$

Let us rewrite the above terms. From the definition of the tangent vector and relation (3.21) we deduce

$$
\begin{aligned}
I =& -\frac{1}{2}\frac{1}{\Delta t}\left|y^{m+1}\right|^2\left(\left|x_{hu}^{m+1}\right|-\left|x_{hu}^m\right|-\left(\tau_h^{m+1},\tau^{m+1}\right)\left|x_{hu}^{m+1}\right|+\left(\tau_h^m,\tau^m\right)\left|x_{hu}^m\right|\right)\\
=& -\frac{1}{4}\frac{1}{\Delta t}\left|y^{m+1}\right|^2\left(\left|\tau^{m+1}-\tau_h^{m+1}\right|^2\left|x_{hu}^{m+1}\right|-\left|\tau^m-\tau_h^m\right|^2\left|x_{hu}^m\right|\right)\\
=& -\frac{1}{4}\frac{1}{\Delta t}\left(\left|y^{m+1}\right|^2\left|\tau^{m+1}-\tau_h^{m+1}\right|^2\left|x_{hu}^{m+1}\right|-\left|y^m\right|^2\left|\tau^m-\tau_h^m\right|^2\left|x_{hu}^m\right|\right)\\
&+\frac{1}{4}\frac{1}{\Delta t}\left(\left|y^{m+1}\right|^2-\left|y^m\right|^2\right)\left|\tau^m-\tau_h^m\right|^2\left|x_{hu}^m\right|.
\end{aligned}
$$

Further, a simple calculation shows

$$
\begin{aligned}
II =& \frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right),\\
III =& \frac{1}{2}\left|y^{m+1}-y_h^{m+1}\right|^2\left(\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right),\\
V =& \left(y^{m+1},y^{m+1}-y_h^{m+1}\right)\left(\tau^{m+1}-\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right).
\end{aligned}
$$

Recalling the definition of the tangent vector and using (3.21), we arrive at

$$
\begin{aligned}
VI =& - \left|y^{m+1}\right|^2 \left(\tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) + \left|y^{m+1}\right|^2 \frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} \\
&- \left(y^{m+1}, y_h^{m+1}\right) \frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} + \left(y^{m+1}, y_h^{m+1}\right) \left(\tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \\
=& - \left|y^{m+1}\right|^2 \frac{|x_u^{m+1}|}{\Delta t} + \left|y^{m+1}\right|^2 \left(\tau^{m+1}, \tau^m\right) \frac{|x_u^m|}{\Delta t} + \left|y^{m+1}\right|^2 \frac{|x_u^{m+1}|}{\Delta t} - \left|y^{m+1}\right|^2 \frac{|x_u^m|}{\Delta t} \\
&- \left(y^{m+1}, y_h^{m+1}\right) \frac{|x_u^{m+1}|}{\Delta t} + \left(y^{m+1}, y_h^{m+1}\right) \frac{|x_u^m|}{\Delta t} \\
&+ \left(y^{m+1}, y_h^{m+1}\right) \frac{|x_u^{m+1}|}{\Delta t} - \left(y^{m+1}, y_h^{m+1}\right) \left(\tau^{m+1}, \tau^m\right) \frac{|x_u^m|}{\Delta t} \\
=& \frac{1}{2} \left(y^{m+1}, y_h^{m+1} - y^{m+1}\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left|x_u^m\right|.
\end{aligned}
$$

Observing the following calculations

$$
\begin{aligned}
\tau^{m+1} - \tau^m &= \tau^{m+1} - \left(\tau^m, \tau^{m+1}\right) \tau^{m+1} - \tau^m + \left(\tau^m, \tau^{m+1}\right) \tau^{m+1} \\
&= \frac{1}{2} \left|\tau^{m+1} - \tau^m\right|^2 \tau^{m+1} - \frac{x_u^m}{|x_u^m|} + \frac{\left(x_u^m, \tau^{m+1}\right) \tau^{m+1}}{|x_u^m|} \\
&= \frac{1}{2} \left|\tau^{m+1} - \tau^m\right|^2 \tau^{m+1} + \frac{x_u^{m+1} - x_u^m}{|x_u^m|} - \frac{\left(x_u^{m+1} - x_u^m, \tau^{m+1}\right) \tau^{m+1}}{|x_u^m|}
\end{aligned}
$$

we rewrite the eighth term as follows

$$
\begin{aligned}
VIII =& - \frac{1}{2} \left|y^{m+1}\right|^2 \left(\tau_h^{m+1}, \frac{\tau^{m+1} - \tau^m}{\Delta t}\right) \left|x_{hu}^m\right| \\
=& - \frac{1}{4} \left|y^{m+1}\right|^2 \left(\tau_h^{m+1}, \tau^{m+1}\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left|x_{hu}^m\right| \\
&- \frac{1}{2} \left|y^{m+1}\right|^2 \left(\tau_h^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \frac{|x_{hu}^m|}{|x_u^m|} \\
&+ \frac{1}{2} \left|y^{m+1}\right|^2 \left(\tau_h^{m+1}, \tau^{m+1}\right) \left(\tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \frac{|x_{hu}^m|}{|x_u^m|}.
\end{aligned}
$$

Further, exploiting (3.21) results in

$$
\begin{aligned}
VIII =& - \frac{1}{4} \left|y^{m+1}\right|^2 \left(\tau_h^{m+1} - \tau^{m+1}, \tau^{m+1}\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left|x_{hu}^m\right| \\
&- \frac{1}{4} \left|y^{m+1}\right|^2 \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left|x_{hu}^m\right| \\
&- \frac{1}{2} \left|y^{m+1}\right|^2 \left(\tau_h^{m+1} - \tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \frac{|x_{hu}^m|}{|x_u^m|} \\
&- \frac{1}{4} \left|y^{m+1}\right|^2 \left|\tau^{m+1} - \tau_h^{m+1}\right|^2 \left(\tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \frac{|x_{hu}^m|}{|x_u^m|}.
\end{aligned} \tag{4.41}
$$

Next, we combine II with the third term on the right-hand side of (4.41) to get

$$\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\left(1-\frac{|x_{hu}^m|}{|x_u^m|}\right).$$

Collecting the above calculations for $I, ..., VIII$ we arrive at

$$\begin{aligned}
A^m =& -\frac{1}{4}\frac{1}{\Delta t}\left(\left|y^{m+1}\right|^2\left|\tau^{m+1}-\tau_h^{m+1}\right|^2\left|x_{hu}^{m+1}\right|-\left|y^m\right|^2\left|\tau^m-\tau_h^m\right|^2\left|x_{hu}^m\right|\right)\\
&+\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\left(1-\frac{|x_{hu}^m|}{|x_u^m|}\right)\\
&+\frac{1}{2}\left|y^{m+1}-y_h^{m+1}\right|^2\left(\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\\
&+\left(y^{m+1},y^{m+1}-y_h^{m+1}\right)\left(\tau^{m+1}-\tau_h^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\\
&+\frac{1}{2}\left(y^{m+1},y_h^{m+1}-y^{m+1}\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left|x_u^m\right|-\frac{1}{4}\left|y_h^{m+1}\right|^2\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t}\left|x_{hu}^m\right|\\
&+\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau_h^m,\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left|x_{hu}^m\right|-\frac{1}{4}\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left|x_{hu}^m\right|\\
&-\frac{1}{4}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau^{m+1},\tau^{m+1}\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left|x_{hu}^m\right|\\
&-\frac{1}{4}\left|y^{m+1}\right|^2\left|\tau^{m+1}-\tau_h^{m+1}\right|^2\left(\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\frac{|x_{hu}^m|}{|x_u^m|}\\
&+\frac{1}{4}\frac{1}{\Delta t}\left(\left|y^{m+1}\right|^2-\left|y^m\right|^2\right)\left|\tau^m-\tau_h^m\right|^2\left|x_{hu}^m\right|\\
=&\sum_{i=1}^{11}A_i.
\end{aligned}$$

Let us consider the terms $A_1, ..., A_{11}$. First, we note that A_1 represents a discrete time derivative and therefore will be included in the difference $\zeta^{m+1}-\zeta^m$.

Let us next estimate $\int_0^{2\pi} A_i$, $(i = 2, .., 11)$ from below, taking advantage of the boundedness of the continuous solution. Thus, the reverse triangle and Young's inequalities along with (3.1) imply

$$\begin{aligned}
\int_0^{2\pi} A_2 =& \frac{1}{2}\int_0^{2\pi}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau^{m+1},\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\left(1-\frac{|x_{hu}^m|}{|x_u^m|}\right)\\
\geq& -C\int_0^{2\pi}\left|y^{m+1}\right|^2\left|\tau_h^{m+1}-\tau^{m+1}\right|\left|\frac{x_u^{m+1}-x_u^m}{\Delta t}\right|\left|\frac{|x_u^m|-|x_{hu}^m|}{|x_u^m|}\right|\\
\geq& -C\left(\left\|\tau^{m+1}-\tau_h^{m+1}\right\|^2+\left\||x_u^m|-|x_{hu}^m|\right\|^2\right).
\end{aligned}\tag{4.42}$$

It is straightforward to see that

$$\int_0^{2\pi} A_3 = \frac{1}{2}\int_0^{2\pi} \left|y^{m+1} - y_h^{m+1}\right|^2 \left(\tau_h^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \geq -C \left\|y^{m+1} - y_h^{m+1}\right\|^2. \quad (4.43)$$

We obtain the following estimate with the help of Cauchy-Schwarz inequality

$$\begin{aligned}\int_0^{2\pi} A_4 &= \int_0^{2\pi} \left(y^{m+1}, y^{m+1} - y_h^{m+1}\right) \left(\tau^{m+1} - \tau_h^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \\ &\geq - C\left(\left\|y^{m+1} - y_h^{m+1}\right\|^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2\right).\end{aligned} \quad (4.44)$$

From the Taylor expansion and boundedness of the continuous solution we infer

$$\begin{aligned}\int_0^{2\pi} A_5 &= \frac{1}{2}\int_0^{2\pi} \left(y^{m+1}, y_h^{m+1} - y^{m+1}\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left|x_u^m\right| \\ &\geq - C\left(\Delta t^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2\right), \\ \int_0^{2\pi} A_9 &= - \frac{1}{4}\int_0^{2\pi} \left|y^{m+1}\right|^2 \left(\tau_h^{m+1} - \tau^{m+1}, \tau^{m+1}\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left|x_{hu}^m\right| \\ &\geq - C\left(\Delta t^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2\right).\end{aligned} \quad (4.45)$$

Analogously, we obtain

$$\begin{aligned}\int_0^{2\pi} A_{10} &= - \frac{1}{4}\int_0^{2\pi} \left|y^{m+1}\right|^2 \left|\tau^{m+1} - \tau_h^{m+1}\right|^2 \left(\tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|} \\ &\geq - C\left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2, \\ \int_0^{2\pi} A_{11} &= \frac{1}{4}\frac{1}{\Delta t}\int_0^{2\pi} \left(\left|y^{m+1}\right|^2 - \left|y^m\right|^2\right) \left|\tau^m - \tau_h^m\right|^2 \left|x_{hu}^m\right| \geq -C \left\|\tau^m - \tau_h^m\right\|^2.\end{aligned} \quad (4.46)$$

In order to combine three remaining terms A_6, A_7 and A_8, one requires the factor $\left|y^{m+1}\right|^2$ in front of A_6. For later use we also change the length element from $|x_{hu}^m|$ to $|x_u^{m+1}|$ for all three terms A_6, A_7 and A_8. To do this, we generate additional terms

$$\begin{aligned}A_6 &= - \frac{1}{4}\left|y_h^{m+1}\right|^2 \frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t} \left|x_{hu}^m\right| \\ &= - \frac{1}{4}\left|y^{m+1}\right|^2 \frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t} \left|x_u^{m+1}\right| + \frac{1}{4}\left(\left|y^{m+1}\right|^2 - \left|y_h^{m+1}\right|^2\right) \frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t} \left|x_{hu}^m\right| \\ &\quad + \frac{1}{4}\left|y^{m+1}\right|^2 \frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t} \left(\left|x_u^{m+1}\right| - \left|x_{hu}^m\right|\right).\end{aligned}$$

Let us denote by α_1 the first term on the right-hand side of the above equation

$$\alpha_1 = -\frac{1}{4}\left|y^{m+1}\right|^2 \frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t} \left|x_u^{m+1}\right|. \quad (4.47)$$

We represent A_6 as $A_6 = \alpha_1 + \tilde{A}_6$, where $\tilde{A}_6$ contains remaining terms. In order to derive the estimate for $\tilde{A}_6$, we recall (3.23) with $a = x_{hu}^{m+1}$ and $b = x_{hu}^m$, which combined with (3.1) produces

$$\left|\tau_h^{m+1} - \tau_h^m\right| \le 2\frac{\left|x_{hu}^{m+1} - x_{hu}^m\right|}{\left|x_{hu}^m\right|} \le \frac{4}{c_0}\left|x_{hu}^{m+1} - x_{hu}^m\right|. \tag{4.48}$$

Recalling next the definition of the error $e_h^m = I_h x^m - x_h^m$ and inserting the Lagrange interpolation operator, we use the triangle inequality and boundedness of the continuous solution to obtain

$$\begin{aligned}\left|\tau_h^{m+1} - \tau_h^m\right| \le & C\left|x_{hu}^{m+1} - x_{hu}^m - \left(I_h\left[x^{m+1} - x^m\right]\right)_u + \left(I_h\left[x^{m+1} - x^m\right]\right)_u\right| \\ \le & C\left(\left|e_{hu}^{m+1} - e_{hu}^m\right| + \Delta t\left|\left(I_h\left[\frac{x^{m+1} - x^m}{\Delta t}\right]\right)_u\right|\right) \\ \le & C\left(\left|e_{hu}^{m+1} - e_{hu}^m\right| + \Delta t\right).\end{aligned} \tag{4.49}$$

Using the above result and the inverse inequality (2.11) we arrive at

$$\left\|\tau_h^{m+1} - \tau_h^m\right\| \le C\Delta t + C\Delta t h^{-1}\left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|. \tag{4.50}$$

Hence, boundedness of the continuous solution and bounds (3.1), (4.16) on the discrete solution together with (4.49) imply

$$\begin{aligned}\int_0^{2\pi} \tilde{A}_6 =& \frac{1}{4}\int_0^{2\pi}\left(\left|y^{m+1}\right|^2 - \left|y_h^{m+1}\right|^2\right)\frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t}\left|x_{hu}^m\right| \\ &+ \frac{1}{4}\int_0^{2\pi}\left|y^{m+1}\right|^2\frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t}\left(\left|x_u^{m+1}\right| - \left|x_{hu}^m\right|\right) \\ \ge& -C\int_0^{2\pi}\left(\left|y^{m+1} - y_h^{m+1}\right| + \left|\left|x_u^{m+1}\right| - \left|x_u^m\right|\right| + \left|\left|x_u^m\right| - \left|x_{hu}^m\right|\right|\right)\frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t} \\ \ge& -C\int_0^{2\pi}\left(\left|y^{m+1} - y_h^{m+1}\right| + \Delta t + \left|\left|x_u^m\right| - \left|x_{hu}^m\right|\right|\right)\left(\frac{\left|e_{hu}^{m+1} - e_{hu}^m\right|^2}{\Delta t} + \Delta t\right) \\ \ge& -C\left(1 + \Delta t\right)\int_0^{2\pi}\frac{\left|e_{hu}^{m+1} - e_{hu}^m\right|^2}{\Delta t} \\ &- C\Delta t\int_0^{2\pi}\left(\left|y^{m+1} - y_h^{m+1}\right| + \Delta t + \left|\left|x_u^m\right| - \left|x_{hu}^m\right|\right|\right).\end{aligned}$$

The inverse estimate (2.11), Cauchy-Schwarz and Young's inequalities further provide

$$\begin{aligned}\int_0^{2\pi}\tilde{A}_6 \ge& -C\left(\Delta t^2 + \left\|\left|x_u^m\right| - \left|x_{hu}^m\right|\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2\right) \\ &- C\left(\Delta t + \Delta t^2\right)h^{-2}\left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2.\end{aligned} \tag{4.51}$$

Introducing

$$\alpha_2 = \frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau_h^m,\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left|x_u^{m+1}\right|, \tag{4.52}$$

A_7 takes the form

$$\begin{aligned}A_7 =&\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau_h^m,\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left|x_{hu}^m\right|\\ =&\alpha_2-\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau_h^m,\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left(\left|x_u^{m+1}\right|-\left|x_{hu}^m\right|\right).\end{aligned}$$

Here we denote the remaining term by $\tilde{A}_7$. Next, with the help of (4.50) we obtain

$$\begin{aligned}\int_0^{2\pi}\tilde{A}_7 =& -\frac{1}{2}\int_0^{2\pi}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau_h^m,\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left(\left|x_u^{m+1}\right|-\left|x_{hu}^m\right|\right)\\ \geq& -C\int_0^{2\pi}\left(\left|\left|x_u^{m+1}\right|-\left|x_u^m\right|\right|+\left|\left|x_u^m\right|-\left|x_{hu}^m\right|\right|\right)\left|\tau_h^{m+1}-\tau_h^m\right|\\ \geq& -C\left(\Delta t^2+\left\|\left|x_u^m\right|-\left|x_{hu}^m\right|\right\|^2+\left(\frac{\Delta t}{h}\right)^2\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2\right).\end{aligned} \tag{4.53}$$

Together with the following notation

$$\alpha_3 = -\frac{1}{4}\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left|x_u^{m+1}\right| \tag{4.54}$$

the term A_8 can be written as

$$A_8 = -\frac{1}{4}\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left|x_{hu}^m\right| = \alpha_3+\frac{1}{4}\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left(\left|x_u^{m+1}\right|-\left|x_{hu}^m\right|\right),$$

where the second term in A_8 we will associate with $\tilde{A}_8$. Hence,

$$\begin{aligned}\int_0^{2\pi}\tilde{A}_8 =& \frac{1}{4}\int_0^{2\pi}\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left(\left|x_u^{m+1}\right|-\left|x_{hu}^m\right|\right)\\ \geq& -C\int_0^{2\pi}\Delta t\left(\left|\left|x_u^{m+1}\right|-\left|x_u^m\right|\right|+\left|\left|x_u^m\right|-\left|x_{hu}^m\right|\right|\right)\\ \geq& -C\left(\Delta t^2+\left\|\left|x_u^m\right|-\left|x_{hu}^m\right|\right\|^2\right).\end{aligned} \tag{4.55}$$

Let us next combine α_1, α_2 and α_3, from (4.47), (4.52) and (4.54), respectively

$$\begin{aligned}&\alpha_1+\alpha_2+\alpha_3 = -\frac{1}{4}\left|y^{m+1}\right|^2\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t}\left|x_u^{m+1}\right|\\ &+\frac{1}{2}\left|y^{m+1}\right|^2\left(\tau_h^{m+1}-\tau_h^m,\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left|x_u^{m+1}\right|-\frac{1}{4}\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left|x_u^{m+1}\right|\\ &= -\frac{1}{4}\frac{1}{\Delta t}\left|y^{m+1}\right|^2\left(\left|\tau_h^{m+1}-\tau_h^m\right|^2-2\left(\tau_h^{m+1}-\tau_h^m,\tau^{m+1}-\tau^m\right)+\left|\tau^{m+1}-\tau^m\right|^2\right)\left|x_u^{m+1}\right|.\end{aligned}$$

Simplifying the above results, yields

$$\alpha := \alpha_1 + \alpha_2 + \alpha_3 = -\frac{1}{4}\frac{1}{\Delta t}\left|y^{m+1}\right|^2\left|\left(\tau^{m+1}-\tau^m\right)-\left(\tau_h^{m+1}-\tau_h^m\right)\right|^2\left|x_u^{m+1}\right|. \tag{4.56}$$

In the next section we combine α with the corresponding term β from B^m.
Finally, collecting the estimates (4.42)-(4.46), (4.51), (4.53) and (4.55) together with (4.56) and recalling the representation of A_1, we obtain

$$\begin{aligned}\int_0^{2\pi} A^m \geq & -\frac{1}{4}\frac{1}{\Delta t}\left(\left|y^{m+1}\right|^2\left|\tau^{m+1}-\tau_h^{m+1}\right|^2\left|x_{hu}^{m+1}\right| - \left|y^m\right|^2\left|\tau^m-\tau_h^m\right|^2\left|x_{hu}^m\right|\right)\\ & -\frac{1}{4}\frac{1}{\Delta t}\int_0^{2\pi}\left|y^{m+1}\right|^2\left|\left(\tau^{m+1}-\tau^m\right)-\left(\tau_h^{m+1}-\tau_h^m\right)\right|^2\left|x_u^{m+1}\right|\\ & - C\Big(\Delta t^2 + \left\|\tau^m-\tau_h^m\right\|^2 + \left\|\tau^{m+1}-\tau_h^{m+1}\right\|^2\\ & \qquad + \left\|\left|x_u^m\right|-\left|x_{hu}^m\right|\right\|^2 + \left\|y^{m+1}-y_h^{m+1}\right\|^2\Big)\\ & - C\frac{\Delta t}{h^2}\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2.\end{aligned} \tag{4.57}$$

Derivation of B^m

Let us now consider the term B^m, defined as follows

$$\begin{aligned}B^m = & \left(\frac{1}{\left|x_u^{m+1}\right|}P^m\left(\frac{x_u^{m+1}-x_u^m}{\Delta t}\right) - \frac{1}{\left|x_{hu}^{m+1}\right|}P_h^m\left(\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right), y_u^{m+1}-y_{hu}^{m+1}\right)\\ & - \left(\frac{1}{\left|x_u^{m+1}\right|}P^m y_u^{m+1} - \frac{1}{\left|x_{hu}^{m+1}\right|}P_h^m y_{hu}^{m+1}, \frac{x_u^{m+1}-x_u^m}{\Delta t} - \frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right).\end{aligned}$$

From the symmetry property of the projection matrix and after simplifying we deduce

$$\begin{aligned}B^m = & \left(y_u^{m+1}, \frac{1}{\left|x_u^{m+1}\right|}P^m\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right) - \left(y_u^{m+1}, \frac{1}{\left|x_{hu}^{m+1}\right|}P_h^m\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right)\\ & + \left(\frac{1}{\left|x_{hu}^{m+1}\right|}P_h^m y_{hu}^{m+1}, \frac{x_u^{m+1}-x_u^m}{\Delta t}\right) - \left(\frac{1}{\left|x_u^{m+1}\right|}P^m y_{hu}^{m+1}, \frac{x_u^{m+1}-x_u^m}{\Delta t}\right).\end{aligned}$$

We recall that we aim to organize terms in B^m in order to obtain cubic differences or discrete time derivatives. To begin, let us rewrite the second term on the right-hand side of the above equation. From (4.32) we infer

$$-\left(y_u^{m+1}, \frac{P_h^m}{\left|x_{hu}^{m+1}\right|}\frac{x_{hu}^{m+1}-x_{hu}^m}{\Delta t}\right) = -\frac{1}{\Delta t}\left(y_u^{m+1}, \tau_h^{m+1}-\tau_h^m\right) - \frac{1}{2}\left(y_u^{m+1}, \tau_h^m\right)\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t}.$$

The above relation and simple calculations lead us to

$$
\begin{aligned}
B^m =& \frac{1}{\Delta t}\left(y_u^{m+1}, \frac{P^{m+1}x_{hu}^{m+1}}{\left|x_u^{m+1}\right|} - \tau_h^{m+1} + \tau^{m+1} - \left(\frac{P^m x_{hu}^m}{\left|x_u^m\right|} - \tau_h^m + \tau^m\right)\right) \\
&- \frac{1}{\Delta t}\left(y_u^{m+1}, \left(\frac{1}{\left|x_u^{m+1}\right|} - \frac{1}{\left|x_u^m\right|}\right) P^m x_{hu}^m\right) - \frac{1}{\Delta t}\left(y_u^{m+1}, \tau^{m+1} - \tau^m\right) \\
&- \frac{1}{\Delta t}\left(y_u^{m+1}, \frac{1}{\left|x_u^{m+1}\right|}\left(P^{m+1} - P^m\right) x_{hu}^{m+1}\right) - \frac{1}{2}\left(y_u^{m+1}, \tau_h^m\right) \frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t} \\
&+ \left(\frac{1}{\left|x_{hu}^{m+1}\right|} P_h^m y_{hu}^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) - \left(\frac{1}{\left|x_u^{m+1}\right|} P^m y_{hu}^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) \\
=& I + \ldots + VII.
\end{aligned}
\tag{4.58}
$$

In order to combine the second, third and fourth terms with the sixth and seventh terms on the right-hand side of (4.58), we need to rewrite II, III and IV in such a way, that a scalar product of the form $(\cdot, x_u^{m+1} - x_u^m)$ appears. To this end, we rewrite the second term with the necessary expression and correct the result with the corresponding difference

$$
\begin{aligned}
II = &-\frac{1}{\Delta t}\left(y_u^{m+1}, -\frac{\left(x_u^{m+1} - x_u^m, \tau^m\right)}{\left|x_u^m\right|\left|x_u^{m+1}\right|} P^m x_{hu}^m\right) \\
&- \frac{1}{\Delta t}\left(y_u^{m+1}, \left(\frac{\left(x_u^{m+1} - x_u^m, \tau^m\right)}{\left|x_u^m\right|\left|x_u^{m+1}\right|} + \left(\frac{1}{\left|x_u^{m+1}\right|} - \frac{1}{\left|x_u^m\right|}\right)\right) P^m x_{hu}^m\right).
\end{aligned}
\tag{4.59}
$$

Representing x_{hu}^m as $\tau_h^m \left|x_{hu}^m\right|$, the first term on the right-hand side of (4.59) takes the form

$$
\begin{aligned}
&\frac{\left(x_u^{m+1} - x_u^m, \tau^m\right)}{\Delta t}\left(y_u^{m+1}, \frac{1}{\left|x_u^m\right|\left|x_u^{m+1}\right|} P^m x_{hu}^m\right) \\
&= \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(P^m \tau_h^m, y_u^{m+1}\right)\tau^m\right).
\end{aligned}
\tag{4.60}
$$

Using (4.32) we derive

$$
\tau^{m+1} - \tau^m = \frac{P^m\left(x_u^{m+1} - x_u^m\right)}{\left|x_u^{m+1}\right|} - \frac{1}{2}\tau^m\left|\tau^{m+1} - \tau^m\right|^2,
\tag{4.61}
$$

what results for the third term in

$$
\begin{aligned}
III &= \left(y_u^{m+1}, \tau^{m+1} - \tau^m\right) \\
&= -\left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \frac{P^m y_u^{m+1}}{\left|x_u^{m+1}\right|}\right) + \frac{1}{2}\left(y_u^{m+1}, \tau^m\right)\frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t}.
\end{aligned}
\tag{4.62}
$$

It remains to rewrite IV. From the definition of the projection matrix we deduce

$$
P^{m+1} - P^m = -\left(\tau^{m+1} - \tau^m\right) \otimes \tau^{m+1} - \tau^m \otimes \left(\tau^{m+1} - \tau^m\right).
\tag{4.63}
$$

If we combine the above relation with (4.61) and $x_{hu}^{m+1} = \tau_h^{m+1}\left|x_{hu}^{m+1}\right|$, we arrive at

$$
\begin{aligned}
IV &= -\frac{1}{\Delta t}\left(y_u^{m+1}, \left(P^{m+1}-P^m\right)\tau_h^{m+1}\right)\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}\\
&= \left(y_u^{m+1}, \frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left(\tau^{m+1}, \tau_h^{m+1}\right)\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}\\
&\quad + \left(y_u^{m+1}, \tau^m\right)\left(\frac{\tau^{m+1}-\tau^m}{\Delta t}, \tau_h^{m+1}\right)\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}\\
&= \left(y_u^{m+1}, \frac{1}{\left|x_u^{m+1}\right|}P^m\frac{x_u^{m+1}-x_u^m}{\Delta t}\right)\left(\tau^{m+1}, \tau_h^{m+1}\right)\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}\\
&\quad + \left(y_u^{m+1}, \tau^m\right)\left(\frac{1}{\left|x_u^{m+1}\right|}P^m\frac{x_u^{m+1}-x_u^m}{\Delta t}, \tau_h^{m+1}\right)\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}\\
&\quad - \frac{1}{2}\left(y_u^{m+1}, \tau^m\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left(\tau^{m+1}+\tau^m, \tau_h^{m+1}\right)\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}.
\end{aligned}
\tag{4.64}
$$

Furthermore, the symmetry property of the projection matrix allows us to write the first and second terms on the right-hand side of (4.64) as

$$
\left(\frac{x_u^{m+1}-x_u^m}{\Delta t}, \left(\tau^{m+1}, \tau_h^{m+1}\right)P^m y_u^{m+1} + \left(y_u^{m+1}, \tau^m\right)P^m\tau_h^{m+1}\right)\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}
\tag{4.65}
$$

In view of (4.59)-(4.62), (4.64) and (4.65) the equation (4.58) takes the form

$$
\begin{aligned}
B^m &= \frac{1}{\Delta t}\left(y_u^{m+1}, \frac{P^{m+1}x_{hu}^{m+1}}{\left|x_u^{m+1}\right|} - \tau_h^{m+1} + \tau^{m+1} - \left(\frac{P^m x_{hu}^m}{\left|x_u^m\right|} - \tau_h^m + \tau^m\right)\right)\\
&\quad + \left(\frac{x_u^{m+1}-x_u^m}{\Delta t}, z_1\right) + \left(\frac{x_u^{m+1}-x_u^m}{\Delta t}, z_2\right)\\
&\quad - \frac{1}{\Delta t}\left(y_u^{m+1}, \left(\frac{\left(x_u^{m+1}-x_u^m, \tau^m\right)}{\left|x_u^m\right|\left|x_u^{m+1}\right|} + \left(\frac{1}{\left|x_u^{m+1}\right|} - \frac{1}{\left|x_u^m\right|}\right)\right)P^m x_{hu}^m\right)\\
&\quad - \frac{1}{2}\left(y_u^{m+1}, \tau_h^m\right)\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t} + \frac{1}{2}\left(y_u^{m+1}, \tau^m\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\\
&\quad - \frac{1}{2}\left(y_u^{m+1}, \tau^m\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\left(\tau^{m+1}+\tau^m, \tau_h^{m+1}\right)\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|},
\end{aligned}
\tag{4.66}
$$

where

$$
\begin{aligned}
z_1 =& \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(\tau^{m+1}, \tau_h^{m+1}\right)P^m y_u^{m+1} - \frac{1}{\left|x_u^{m+1}\right|}P^m y_u^{m+1} - \frac{1}{\left|x_u^{m+1}\right|}P^m y_{hu}^{m+1} + \frac{1}{\left|x_{hu}^{m+1}\right|}P^m y_{hu}^{m+1},\\
z_2 =& \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(P^m\tau_h^m, y_u^{m+1}\right)\tau^m + \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(y_u^{m+1}, \tau^m\right)P^m\tau_h^{m+1} + \frac{1}{\left|x_{hu}^{m+1}\right|}P_h^m y_{hu}^{m+1}\\
&- \frac{1}{\left|x_{hu}^{m+1}\right|}P^m y_{hu}^{m+1}.
\end{aligned}
$$

Let us rewrite z_1 and z_2. From (3.21) follows

$$
\begin{aligned}
z_1 =& \frac{1}{\left|x_{hu}^{m+1}\right|}\left(\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}-1\right)^2 P^m y_u^{m+1} - \frac{1}{2}\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left|\tau^{m+1}-\tau_h^{m+1}\right|^2 P^m y_u^{m+1} \\
&+ \left(\frac{1}{\left|x_u^{m+1}\right|}-\frac{1}{\left|x_{hu}^{m+1}\right|}\right)\left(P^m\left(y_u^{m+1}-y_{hu}^{m+1}\right)\right).
\end{aligned}
$$

Now we turn our attention to z_2. From the definition of the projection matrix we deduce

$$
\begin{aligned}
z_2 =& \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(\tau_h^m, y_u^{m+1}\right)\tau^m - \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(\tau_h^m,\tau^m\right)\left(\tau^m, y_u^{m+1}\right)\tau^m \\
&+ \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(y_u^{m+1},\tau^m\right)\tau_h^{m+1} - \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(y_u^{m+1},\tau^m\right)\left(\tau^m,\tau_h^{m+1}\right)\tau^m \\
&+ \frac{1}{\left|x_{hu}^{m+1}\right|}y_{hu}^{m+1} - \frac{1}{\left|x_{hu}^{m+1}\right|}\left(y_{hu}^{m+1},\tau_h^m\right)\tau_h^m - \frac{1}{\left|x_{hu}^{m+1}\right|}y_{hu}^{m+1} + \frac{1}{\left|x_{hu}^{m+1}\right|}\left(y_{hu}^{m+1},\tau^m\right)\tau^m
\end{aligned}
$$

and observe the cancellation of the fifth and seventh terms. Next, we add zero terms to z_2 and combine the resulting expressions in the following way

$$
\begin{aligned}
z_2 =& \left(\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau_h^m\right)\tau^m - \frac{1}{\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau_h^m\right)\tau_h^m\right. \\
&\left. - \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\tau^m + \frac{1}{\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\tau_h^m\right) \\
&+ \left(\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(y_u^{m+1},\tau^m\right)\tau_h^{m+1} - \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(y_u^{m+1},\tau^m\right)\tau^m\right. \\
&\left. - \frac{1}{\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\tau_h^{m+1} + \frac{1}{\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\tau^m\right) \\
&+ \left(\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\tau^m - \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\left(\tau^m,\tau_h^m\right)\tau^m\right) \\
&+ \left(\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(y_u^{m+1},\tau^m\right)\tau^m - \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(y_u^{m+1},\tau^m\right)\left(\tau^m,\tau_h^{m+1}\right)\tau^m\right) \\
&+ \left(\frac{1}{\left|x_{hu}^{m+1}\right|}\left(y_{hu}^{m+1},\tau^m\right)\tau^m - \frac{1}{\left|x_{hu}^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\tau^m\right. \\
&\left. - \frac{1}{\left|x_{hu}^{m+1}\right|}\left(y_{hu}^{m+1},\tau_h^m\right)\tau_h^m + \frac{1}{\left|x_{hu}^{m+1}\right|}\left(y_u^{m+1},\tau_h^m\right)\tau_h^m\right) \\
&+ \left(\frac{1}{\left|x_{hu}^{m+1}\right|}-\frac{1}{\left|x_u^{m+1}\right|}\right)\left(\left(y_u^{m+1},\tau^m\right)\tau^m - \left(y_u^{m+1},\tau_h^m\right)\tau_h^m\right) \\
&+ \frac{1}{\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\left(\tau_h^{m+1}-\tau_h^m\right).
\end{aligned}
$$

Using now (3.21), we can rewrite z_2 as

$$
\begin{aligned}
z_2 = &\left(y_u^{m+1}, \tau_h^m - \tau^m\right) \frac{1}{|x_u^{m+1}|} \left(\frac{|x_{hu}^m|}{|x_u^m|}\tau^m - \tau_h^m\right) \\
&+ \left(y_u^{m+1}, \tau^m\right) \frac{1}{|x_u^{m+1}|} \left(\frac{\left|x_{hu}^{m+1}\right|}{|x_u^{m+1}|} - 1\right) \left(\tau_h^{m+1} - \tau^m\right) \\
&+ \frac{1}{2} \frac{|x_{hu}^m|}{|x_u^m|\,|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m\right) |\tau^m - \tau_h^m|^2 \,\tau^m \\
&+ \frac{1}{2} \frac{\left|x_{hu}^{m+1}\right|}{|x_u^{m+1}|^2} \left(y_u^{m+1}, \tau^m\right) \left|\tau^m - \tau_h^{m+1}\right|^2 \tau^m \\
&+ \frac{1}{\left|x_{hu}^{m+1}\right|} \left(\left(y_{hu}^{m+1} - y_u^{m+1}, \tau^m\right) \tau^m - \left(y_{hu}^{m+1} - y_u^{m+1}, \tau_h^m\right) \tau_h^m\right) \\
&+ \left(\frac{1}{\left|x_{hu}^{m+1}\right|} - \frac{1}{|x_u^{m+1}|}\right) \left(\left(y_u^{m+1}, \tau^m\right) \tau^m - \left(y_u^{m+1}, \tau_h^m\right) \tau_h^m\right) \\
&+ \frac{1}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m\right) \left(\tau_h^{m+1} - \tau_h^m\right) .
\end{aligned}
$$

Together with the following notation

$$
\tilde{z}_2 = z_2 - \frac{1}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m\right) \left(\tau_h^{m+1} - \tau_h^m\right)
$$

equation (4.66) translates into

$$
\begin{aligned}
B^m = &\frac{1}{\Delta t} \left(y_u^{m+1}, \frac{1}{|x_u^{m+1}|} P^{m+1} x_{hu}^{m+1} - \tau_h^{m+1} + \tau^{m+1} - \left(\frac{1}{|x_u^m|} P^m x_{hu}^m - \tau_h^m + \tau^m\right)\right) \\
&+ \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, z_1\right) + \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tilde{z}_2\right) \\
&+ \frac{1}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m\right) \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tau_h^{m+1} - \tau_h^m\right) \\
&- \frac{1}{\Delta t} \left(y_u^{m+1}, \left(\frac{\left(x_u^{m+1} - x_u^m, \tau^m\right)}{|x_u^m|\,|x_u^{m+1}|} + \left(\frac{1}{|x_u^{m+1}|} - \frac{1}{|x_u^m|}\right)\right) P^m x_{hu}^m\right) \\
&- \frac{1}{2} \left(y_u^{m+1}, \tau_h^m\right) \frac{\left|\tau_h^{m+1} - \tau_h^m\right|^2}{\Delta t} + \frac{1}{2} \left(y_u^{m+1}, \tau^m\right) \frac{|\tau^{m+1} - \tau^m|^2}{\Delta t} \\
&- \frac{1}{2} \left(y_u^{m+1}, \tau^m\right) \frac{|\tau^{m+1} - \tau^m|^2}{\Delta t} \left(\tau^{m+1} + \tau^m, \tau_h^{m+1}\right) \frac{\left|x_{hu}^{m+1}\right|}{|x_u^{m+1}|} \\
= &\sum_{i=1}^{8} B_i,
\end{aligned}
$$

where z_1 is given by

$$\begin{aligned}
z_1 =& \frac{1}{\left|x_{hu}^{m+1}\right|}\left(\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}-1\right)^2 P^m y_u^{m+1} - \frac{1}{2}\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left|\tau^{m+1}-\tau_h^{m+1}\right|^2 P^m y_u^{m+1} \\
&+\left(\frac{1}{\left|x_u^{m+1}\right|}-\frac{1}{\left|x_{hu}^{m+1}\right|}\right)\left(P^m\left(y_u^{m+1}-y_{hu}^{m+1}\right)\right)
\end{aligned} \tag{4.67}$$

and z_2 takes the form

$$\begin{aligned}
\tilde{z}_2 =& \left(y_u^{m+1}, \tau_h^m - \tau^m\right)\left(\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\tau^m - \frac{1}{\left|x_u^{m+1}\right|}\tau_h^m\right) \\
&+\left(y_u^{m+1},\tau^m\right)\frac{1}{\left|x_u^{m+1}\right|}\left(\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}-1\right)\left(\tau_h^{m+1}-\tau^m\right) \\
&+\frac{1}{2}\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|\left|x_u^{m+1}\right|}\left(y_u^{m+1},\tau^m\right)\left|\tau^m-\tau_h^m\right|^2\tau^m \\
&+\frac{1}{2}\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\left(y_u^{m+1},\tau^m\right)\left|\tau^m-\tau_h^{m+1}\right|^2\tau^m \\
&+\frac{1}{\left|x_{hu}^{m+1}\right|}\left(\left(y_{hu}^{m+1}-y_u^{m+1},\tau^m\right)\tau^m - \left(y_{hu}^{m+1}-y_u^{m+1},\tau_h^m\right)\tau_h^m\right) \\
&+\left(\frac{1}{\left|x_{hu}^{m+1}\right|}-\frac{1}{\left|x_u^{m+1}\right|}\right)\left(\left(y_u^{m+1},\tau^m\right)\tau^m - \left(y_u^{m+1},\tau_h^m\right)\tau_h^m\right).
\end{aligned} \tag{4.68}$$

Let us consider B_1 in order to produce discrete time derivatives. To begin, we split it into two terms

$$\begin{aligned}
B_1 =& \frac{1}{\Delta t}\left(y_u^{m+1}, \frac{1}{\left|x_u^{m+1}\right|}P^{m+1}x_{hu}^{m+1} - \tau_h^{m+1} + \tau^{m+1} - \left(\frac{1}{\left|x_u^m\right|}P^m x_{hu}^m - \tau_h^m + \tau^m\right)\right) \\
=& \frac{1}{\Delta t}\left(\left(y_u^{m+1}, \frac{1}{\left|x_u^{m+1}\right|}P^{m+1}x_{hu}^{m+1} - \tau_h^{m+1} + \tau^{m+1}\right) - \left(y_u^m, \frac{1}{\left|x_u^m\right|}P^m x_{hu}^m - \tau_h^m + \tau^m\right)\right) \\
&- \frac{1}{\Delta t}\left(y_u^{m+1} - y_u^m, \frac{1}{\left|x_u^m\right|}P^m x_{hu}^m - \tau_h^m + \tau^m\right) \\
=& B_{1,1} + B_{1,2}.
\end{aligned}$$

Our aim now is to rewrite $B_{1,1}$ and $B_{1,2}$ in such a way that they contain 3 differences. Hence, from the definition of the projection matrix and (3.21) we deduce

$$\begin{aligned}
\frac{1}{\left|x_u^m\right|}P^m x_{hu}^m - \tau_h^m + \tau^m =& \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|}\tau_h^m - \frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|}\left(\tau^m,\tau_h^m\right)\tau^m - \tau_h^m + \tau^m \\
=& \left(\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|}-1\right)\left(\tau_h^m-\tau^m\right) + \frac{1}{2}\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|}\left|\tau_h^m-\tau^m\right|^2\tau^m.
\end{aligned}$$

Exploiting the above relation, we may continue with

$$
\begin{aligned}
B_{1,1} =& \frac{1}{\Delta t}\left(y_u^{m+1}, \left(\frac{|x_{hu}^{m+1}|}{|x_u^{m+1}|} - 1\right)\left(\tau_h^{m+1} - \tau^{m+1}\right)\right. \\
&\left. + \frac{1}{2}\frac{|x_{hu}^{m+1}|}{|x_u^{m+1}|}\left|\tau_h^{m+1} - \tau^{m+1}\right|^2 \tau^{m+1}\right) \\
& - \frac{1}{\Delta t}\left(y_u^m, \left(\frac{|x_{hu}^m|}{|x_u^m|} - 1\right)\left(\tau_h^m - \tau^m\right) + \frac{1}{2}\frac{|x_{hu}^m|}{|x_u^m|}\left|\tau_h^m - \tau^m\right|^2 \tau^m\right), \\
B_{1,2} =& -\frac{1}{\Delta t}\left(y_u^{m+1} - y_u^m, \left(\frac{|x_{hu}^m|}{|x_u^m|} - 1\right)\left(\tau_h^m - \tau^m\right) + \frac{1}{2}\frac{|x_{hu}^m|}{|x_u^m|}\left|\tau_h^m - \tau^m\right|^2 \tau^m\right).
\end{aligned}
\tag{4.69}
$$

We note that $B_{1,1}$ is a part of the difference $\zeta^{m+1} - \zeta^m$. On the other hand, terms which are not included in this difference have to be estimated. Hence, from the boundedness of the continuous solution, (4.16), Cauchy-Schwarz and Young's inequalities we infer

$$
\begin{aligned}
\int_0^{2\pi} B_{1,2} =& -\int_0^{2\pi}\left(\frac{y_u^{m+1} - y_u^m}{\Delta t}, \left(\frac{|x_{hu}^m|}{|x_u^m|} - 1\right)\left(\tau_h^m - \tau^m\right) + \frac{1}{2}\frac{|x_{hu}^m|}{|x_u^m|}\left|\tau_h^m - \tau^m\right|^2 \tau^m\right) \\
\geq& -C\int_0^{2\pi}\frac{\left||x_{hu}^m| - |x_u^m|\right|}{|x_u^m|}\left|\tau_h^m - \tau^m\right| - C\int_0^{2\pi}\frac{|x_{hu}^m|}{|x_u^m|}\left|\tau_h^m - \tau^m\right|^2\left|\tau^m\right| \\
\geq& -C\left(\left\||x_u^m| - |x_{hu}^m|\right\|^2 + \left\|\tau^m - \tau_h^m\right\|^2\right).
\end{aligned}
$$

Let us estimate $\int_0^{2\pi} B_2 = \int_0^{2\pi}\left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, z_1\right)$, where z_1 is given by (4.67). Using the boundedness of the continuous solution and Young's inequality we deduce

$$
\begin{aligned}
\int_0^{2\pi} B_2 =& \int_0^{2\pi}\left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \frac{1}{|x_{hu}^{m+1}|}\left(\frac{|x_{hu}^{m+1}|}{|x_u^{m+1}|} - 1\right)^2 P^m y_u^{m+1}\right) \\
& - \frac{1}{2}\int_0^{2\pi}\left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \frac{|x_{hu}^{m+1}|}{|x_u^{m+1}|^2}\left|\tau^{m+1} - \tau_h^{m+1}\right|^2 P^m y_u^{m+1}\right) \\
& + \int_0^{2\pi}\left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \left(\frac{1}{|x_u^{m+1}|} - \frac{1}{|x_{hu}^{m+1}|}\right)\left(P^m\left(y_u^{m+1} - y_{hu}^{m+1}\right)\right)\right) \\
\geq& -C\left(\left\||x_u^{m+1}| - |x_{hu}^{m+1}|\right\|^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2 + \left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2\right).
\end{aligned}
$$

We turn now our attention to B_3, where $B_3 = \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tilde{z}_2\right)$ with $\tilde{z}_2$ given by (4.68).

For practical reasons, we enumerate the integrals

$$
\begin{aligned}
\int_0^{2\pi} B_3 = &\int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \frac{|x_{hu}^m|}{|x_u^m|\,|x_u^{m+1}|}\tau^m - \frac{1}{|x_u^{m+1}|}\tau_h^m \right) \left(y_u^{m+1}, \tau_h^m - \tau^m\right) \\
&+ \int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tau_h^{m+1} - \tau^m \right) \frac{1}{|x_u^{m+1}|} \left(\frac{\left|x_{hu}^{m+1}\right|}{|x_u^{m+1}|} - 1 \right) \left(y_u^{m+1}, \tau^m\right) \\
&+ \frac{1}{2} \int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tau^m \right) \left(y_u^{m+1}, \tau^m\right) |\tau^m - \tau_h^m|^2 \frac{|x_{hu}^m|}{|x_u^m|\,|x_u^{m+1}|} \\
&+ \frac{1}{2} \int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tau^m \right) \left(y_u^{m+1}, \tau^m\right) \left|\tau^m - \tau_h^{m+1}\right|^2 \frac{\left|x_{hu}^{m+1}\right|}{|x_u^{m+1}|^2} \\
&+ \int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \left(y_{hu}^{m+1} - y_u^{m+1}, \tau^m\right)\tau^m - \left(y_{hu}^{m+1} - y_u^{m+1}, \tau_h^m\right)\tau_h^m \right) \frac{1}{\left|x_{hu}^{m+1}\right|} \\
&+ \int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \left(y_u^{m+1}, \tau^m\right)\tau^m - \left(y_u^{m+1}, \tau_h^m\right)\tau_h^m \right) \left(\frac{1}{\left|x_{hu}^{m+1}\right|} - \frac{1}{|x_u^{m+1}|} \right) \\
= &\sum_{i=0}^{6} B_{3,i}.
\end{aligned}
$$

We consider the integrals separately, using the boundedness of the continuous solution. From Cauchy-Schwarz and Young's inequality, (4.1) together with (4.16) follows

$$
\begin{aligned}
B_{3,1} = &\int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \frac{1}{|x_u^{m+1}|}\tau^m \left(\frac{|x_{hu}^m|}{|x_u^m|} - 1 \right) \right) \left(y_u^{m+1}, \tau_h^m - \tau^m\right) \\
&+ \int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \frac{1}{|x_u^{m+1}|}\left(\tau^m - \tau_h^m\right) \right) \left(y_u^{m+1}, \tau_h^m - \tau^m\right) \\
\geq &- C \left(\||x_u^m| - |x_{hu}^m|\|^2 + \|\tau^m - \tau_h^m\|^2 \right).
\end{aligned}
$$

Using similar arguments and Taylor expansion $\tau^{m+1} - \tau^m = \tau_t^m \Delta t + O(\Delta t^2)$ we obtain

$$
\begin{aligned}
B_{3,2} = &\int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tau_h^{m+1} - \tau^{m+1} \right) \frac{1}{|x_u^{m+1}|} \left(\frac{\left|x_{hu}^{m+1}\right|}{|x_u^{m+1}|} - 1 \right) \left(y_u^{m+1}, \tau^m\right) \\
&+ \int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tau^{m+1} - \tau^m \right) \frac{1}{|x_u^{m+1}|} \left(\frac{\left|x_{hu}^{m+1}\right|}{|x_u^{m+1}|} - 1 \right) \left(y_u^{m+1}, \tau^m\right) \\
\geq &- C \left(\Delta t^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2 + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 \right).
\end{aligned}
$$

Furthermore, it is straightforward to see that

$$
B_{3,3} = \frac{1}{2} \int_0^{2\pi} \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tau^m \right) \left(y_u^{m+1}, \tau^m\right) |\tau^m - \tau_h^m|^2 \frac{|x_{hu}^m|}{|x_u^m|\,|x_u^{m+1}|} \geq -C \|\tau^m - \tau_h^m\|^2 .
$$

With the help of the triangle inequality we derive an estimate

$$\begin{aligned}B_{3,4} =&\frac{1}{2}\int_0^{2\pi}\left(\frac{x_u^{m+1}-x_u^m}{\Delta t},\tau^m\right)\left(y_u^{m+1},\tau^m\right)\left|\tau^m-\tau_h^{m+1}\right|^2\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|^2}\\ \geq&-C\left(\Delta t^2+\left\|\tau^{m+1}-\tau_h^{m+1}\right\|^2\right).\end{aligned}$$

For the next integral we have

$$\begin{aligned}B_{3,5} =&\int_0^{2\pi}\left(\frac{x_u^{m+1}-x_u^m}{\Delta t},\tau^m-\tau_h^m\right)\left(y_{hu}^{m+1}-y_u^{m+1},\tau^m\right)\frac{1}{\left|x_{hu}^{m+1}\right|}\\ &+\int_0^{2\pi}\left(\frac{x_u^{m+1}-x_u^m}{\Delta t},\tau_h^m\right)\left(y_{hu}^{m+1}-y_u^{m+1},\tau^m-\tau_h^m\right)\frac{1}{\left|x_{hu}^{m+1}\right|}\\ \geq&-C\left(\left\|\tau^m-\tau_h^m\right\|^2+\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|^2\right).\end{aligned}$$

Similarly as above, we obtain

$$\begin{aligned}B_{3,6} =&\int_0^{2\pi}\left(\frac{x_u^{m+1}-x_u^m}{\Delta t},\tau^m-\tau_h^m\right)\left(y_u^{m+1},\tau^m\right)\left(\frac{1}{\left|x_{hu}^{m+1}\right|}-\frac{1}{\left|x_u^{m+1}\right|}\right)\\ &+\int_0^{2\pi}\left(\frac{x_u^{m+1}-x_u^m}{\Delta t},\tau_h^m\right)\left(y_u^{m+1},\tau^m-\tau_h^m\right)\left(\frac{1}{\left|x_{hu}^{m+1}\right|}-\frac{1}{\left|x_u^{m+1}\right|}\right)\\ \geq&-C\left(\left\|\tau^m-\tau_h^m\right\|^2+\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|^2\right).\end{aligned}$$

Observing that

$$\begin{aligned}-\frac{\left(x_u^{m+1}-x_u^m,\tau^m\right)}{\left|x_u^m\right|\left|x_u^{m+1}\right|}-\left(\frac{1}{\left|x_u^{m+1}\right|}-\frac{1}{\left|x_u^m\right|}\right)=&-\frac{\left(\tau^{m+1},\tau^m\right)}{\left|x_u^m\right|}+\frac{\left(\tau^m,\tau^m\right)}{\left|x_u^{m+1}\right|}-\frac{1}{\left|x_u^{m+1}\right|}+\frac{1}{\left|x_u^m\right|}\\ =&\frac{1}{2}\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\left|x_u^m\right|},\end{aligned}$$

and recalling the definition of the projection matrix the fifth term can be rewritten as

$$\begin{aligned}B_5 =&-\frac{1}{\Delta t}\left(y_u^{m+1},\left(\frac{\left(x_u^{m+1}-x_u^m,\tau^m\right)}{\left|x_u^m\right|\left|x_u^{m+1}\right|}+\left(\frac{1}{\left|x_u^{m+1}\right|}-\frac{1}{\left|x_u^m\right|}\right)\right)P^m x_{hu}^m\right)\\ =&\frac{1}{2}\frac{1}{\Delta t}\left(y_u^{m+1},P^m x_{hu}^m\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\left|x_u^m\right|}\\ =&\frac{1}{2}\frac{1}{\Delta t}\left(y_u^{m+1},\tau_h^m\right)\left|\tau^{m+1}-\tau^m\right|^2\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|}-\frac{1}{2}\frac{1}{\Delta t}\left(y_u^{m+1},\tau^m\right)\left(\tau^m,\tau_h^m\right)\left|\tau^{m+1}-\tau^m\right|^2\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|}.\end{aligned}$$

Using additionally (3.21) leads to

$$\begin{aligned}B_5 =&\frac{1}{2}\frac{1}{\Delta t}\left(y_u^{m+1},\tau_h^m-\tau^m\right)\left|\tau^{m+1}-\tau^m\right|^2\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|}\\ &+\frac{1}{4}\frac{1}{\Delta t}\left(y_u^{m+1},\tau^m\right)\left|\tau^m-\tau_h^m\right|^2\left|\tau^{m+1}-\tau^m\right|^2\frac{\left|x_{hu}^m\right|}{\left|x_u^m\right|}.\end{aligned}$$

A similar argument as above may be used to obtain

$$\int_0^{2\pi} B_5 \geq -C \int_0^{2\pi} \Delta t \left(|\tau^m - \tau_h^m| + |\tau^m - \tau_h^m|^2 \right) \geq -C \left(\Delta t^2 + \|\tau^m - \tau_h^m\|^2 \right).$$

Let us next consider B_4. With the help of (3.21) we infer

$$\begin{aligned}
B_4 =& \frac{1}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m \right) \left(\frac{x_u^{m+1} - x_u^m}{\Delta t}, \tau_h^{m+1} - \tau_h^m \right) \\
=& \frac{1}{\Delta t} \left(y_u^{m+1}, \tau^m \right) \left(\tau^{m+1}, \tau_h^{m+1} \right) - \frac{1}{\Delta t} \left(y_u^{m+1}, \tau^m \right) \left(\tau^{m+1}, \tau_h^m \right) \\
& - \frac{1}{\Delta t} \frac{|x_u^m|}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m \right) \left(\tau^m, \tau_h^{m+1} \right) + \frac{1}{\Delta t} \frac{|x_u^m|}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m \right) \left(\tau^m, \tau_h^m \right) \\
=& - \frac{1}{2} \frac{1}{\Delta t} \left(y_u^{m+1}, \tau^m \right) \left| \tau^{m+1} - \tau_h^{m+1} \right|^2 + \frac{1}{2} \frac{1}{\Delta t} \left(y_u^{m+1}, \tau^m \right) \left| \tau^{m+1} - \tau_h^m \right|^2 \\
& + \frac{1}{2} \frac{1}{\Delta t} \frac{|x_u^m|}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m \right) \left| \tau^m - \tau_h^{m+1} \right|^2 - \frac{1}{2} \frac{1}{\Delta t} \frac{|x_u^m|}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m \right) |\tau^m - \tau_h^m|^2 .
\end{aligned}$$

To proceed, we note that τ^m is orthogonal to τ_u^m as a vector of constant length. Recalling further the definition of the curvature vector $y^m = \frac{1}{|x_u^m|} \tau_u^m$ we observe

$$(y_u^m, \tau^m) = (y^m, \tau^m)_u - (y^m, \tau_u^m) = -(y^m, \tau_u^m) = -|x_u^m| \, |y^m|^2, \quad m \in [0, ..., M]. \tag{4.70}$$

Using (4.70) we rewrite the scalar product (y_u^{m+1}, τ^m) as

$$\begin{aligned}
\left(y_u^{m+1}, \tau^m \right) &= \left(y_u^{m+1}, \tau^m - \tau^{m+1} \right) + \left(y_u^{m+1}, \tau^{m+1} \right) \\
&= \left(y_u^{m+1}, \tau^m - \tau^{m+1} \right) - \left| x_u^{m+1} \right| \left| y^{m+1} \right|^2 .
\end{aligned} \tag{4.71}$$

Next, (4.71) allows us to write B_4 as

$$\begin{aligned}
B_4 =& - \frac{1}{2} \left(y_u^{m+1}, \tau^m - \tau^{m+1} \right) \frac{\left| \tau^{m+1} - \tau_h^{m+1} \right|^2}{\Delta t} + \frac{1}{2} \left| x_u^{m+1} \right| \left| y^{m+1} \right|^2 \frac{\left| \tau^{m+1} - \tau_h^{m+1} \right|^2}{\Delta t} \\
& + \frac{1}{2} \left(y_u^{m+1}, \tau^m - \tau^{m+1} \right) \frac{|\tau^{m+1} - \tau_h^m|^2}{\Delta t} - \frac{1}{2} \left| x_u^{m+1} \right| \left| y^{m+1} \right|^2 \frac{|\tau^{m+1} - \tau_h^m|^2}{\Delta t} \\
& + \frac{1}{2} \frac{|x_u^m|}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m - \tau^{m+1} \right) \frac{\left| \tau^m - \tau_h^{m+1} \right|^2}{\Delta t} - \frac{1}{2} |x_u^m| \left| y^{m+1} \right|^2 \frac{\left| \tau^m - \tau_h^{m+1} \right|^2}{\Delta t} \\
& - \frac{1}{2} \frac{|x_u^m|}{|x_u^{m+1}|} \left(y_u^{m+1}, \tau^m - \tau^{m+1} \right) \frac{|\tau^m - \tau_h^m|^2}{\Delta t} + \frac{1}{2} |x_u^m| \left| y^{m+1} \right|^2 \frac{|\tau^m - \tau_h^m|^2}{\Delta t}.
\end{aligned}$$

For simplicity, we represent B_4 as

$$B_4 = \beta_1 + \tilde{B}_4,$$

where

$$\begin{aligned}\beta_1 =& \frac{1}{2}\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau_h^{m+1}\right|^2}{\Delta t}-\frac{1}{2}\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau_h^{m}\right|^2}{\Delta t}\\ &-\frac{1}{2}\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2\frac{\left|\tau^{m}-\tau_h^{m+1}\right|^2}{\Delta t}+\frac{1}{2}\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2\frac{\left|\tau^{m}-\tau_h^{m}\right|^2}{\Delta t}\end{aligned} \tag{4.72}$$

and

$$\begin{aligned}\tilde{B}_4 =& \frac{1}{2}\left(y_u^{m+1},\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left(\left|\tau^{m+1}-\tau_h^{m+1}\right|^2-\left|\tau^{m+1}-\tau_h^{m}\right|^2\right)\\ &-\frac{1}{2}\frac{\left|x_u^m\right|}{\left|x_u^{m+1}\right|}\left(y_u^{m+1},\frac{\tau^{m+1}-\tau^m}{\Delta t}\right)\left(\left|\tau^{m}-\tau_h^{m+1}\right|^2-\left|\tau^{m}-\tau_h^{m}\right|^2\right)\\ &+\frac{1}{2}\frac{\left|x_u^{m+1}\right|-\left|x_u^m\right|}{\Delta t}\left|y^{m+1}\right|^2\left(\left|\tau^{m}-\tau_h^{m+1}\right|^2-\left|\tau^{m}-\tau_h^{m}\right|^2\right).\end{aligned}$$

In view of the boundedness of the continuous solution, Young's and reverse triangle inequalities, we deduce

$$\int_0^{2\pi}\tilde{B}_4 \geq -C\left(\Delta t^2+\|\tau^m-\tau_h^m\|^2+\left\|\tau^{m+1}-\tau_h^{m+1}\right\|^2\right).$$

For B_6 and B_7 we analogously obtain

$$\begin{aligned}B_6 =& -\frac{1}{2}\left(y_u^{m+1},\tau_h^m\right)\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t}\\ =& -\frac{1}{2}\left(y_u^{m+1},\tau_h^m-\tau^{m+1}\right)\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t}+\frac{1}{2}\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t},\\ B_7 =& \frac{1}{2}\left(y_u^{m+1},\tau^m\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}\\ =& \frac{1}{2}\left(y_u^{m+1},\tau^m-\tau^{m+1}\right)\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}-\frac{1}{2}\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t}.\end{aligned}$$

If we denote by

$$\begin{aligned}\beta_2 =& \frac{1}{2}\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t},\\ \beta_3 =& -\frac{1}{2}\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2\frac{\left|\tau^{m+1}-\tau^m\right|^2}{\Delta t},\end{aligned} \tag{4.73}$$

B_6 and B_7 can be written as $B_6=\beta_2+\tilde{B}_6$ and $B_7=\beta_3+\tilde{B}_7$, respectively. Here, $\tilde{B}_6$ and $\tilde{B}_7$ contain the first term in B_6 and B_7, respectively. Analogously to (4.51), we derive

$$\begin{aligned}\int_0^{2\pi}\tilde{B}_6 =& -\frac{1}{2}\int_0^{2\pi}\left(y_u^{m+1},\tau_h^m-\tau^{m+1}\right)\frac{\left|\tau_h^{m+1}-\tau_h^m\right|^2}{\Delta t}\\ \geq& -C\left(\Delta t^2+\|\tau^m-\tau_h^m\|^2+\Delta t h^{-2}\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2\right).\end{aligned}$$

From the boundedness of the continuous solution we infer

$$\int_0^{2\pi} \tilde{B}_7 = \int_0^{2\pi} \frac{1}{2} \left(y_u^{m+1}, \tau^m - \tau^{m+1}\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \geq -C\Delta t^2.$$

Following the ideas above and using the relations (4.70), (4.71) we get

$$\begin{aligned}
B_8 =& - \frac{1}{2} \left(y_u^{m+1}, \tau^m\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left(\tau^{m+1} + \tau^m, \tau_h^{m+1}\right) \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|} \\
=& - \frac{1}{2} \left(y_u^{m+1}, \tau^m\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left(\tau^{m+1} - \tau_h^{m+1}, \tau_h^{m+1}\right) \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|} \\
& - \frac{1}{2} \left(y_u^{m+1}, \tau^m\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left(\tau^m - \tau_h^{m+1}, \tau_h^{m+1}\right) \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|} \\
& - \left(y_u^{m+1}, \tau^m\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|} \\
=& - \frac{1}{2} \left(y_u^{m+1}, \tau^m\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left(\tau^{m+1} - \tau_h^{m+1}, \tau_h^{m+1}\right) \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|} \\
& - \frac{1}{2} \left(y_u^{m+1}, \tau^m\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left(\tau^m - \tau_h^{m+1}, \tau_h^{m+1}\right) \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|} \\
& - \left(y_u^{m+1}, \tau^m - \tau^{m+1}\right) \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|} + \left|y^{m+1}\right|^2 \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left|x_{hu}^{m+1}\right|.
\end{aligned}$$

Denoting by

$$\beta_4 = \left|y^{m+1}\right|^2 \frac{\left|\tau^{m+1} - \tau^m\right|^2}{\Delta t} \left|x_u^{m+1}\right| \tag{4.74}$$

and by $\tilde{B}_8$ the remaining in B_8 terms we use Young's inequality and (4.16) to obtain

$$\int_0^{2\pi} \tilde{B}_8 \geq - C \left(\Delta t^2 + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2\right).$$

Let us consider the sum of $\beta_1, ..., \beta_4$ given by (4.72), (4.73) and (4.74), respectively

$$\begin{aligned}
\beta :=& \beta_1 + \beta_2 + \beta_3 + \beta_4 = \frac{1}{2} \frac{1}{\Delta t} \left|x_u^{m+1}\right| \left|y^{m+1}\right|^2 \left|\tau^{m+1} - \tau_h^{m+1}\right|^2 \\
& - \frac{1}{2} \frac{1}{\Delta t} \left|x_u^{m+1}\right| \left|y^{m+1}\right|^2 \left|\tau^{m+1} - \tau_h^m\right|^2 - \frac{1}{2} \frac{1}{\Delta t} \left|x_u^{m+1}\right| \left|y^{m+1}\right|^2 \left|\tau^m - \tau_h^{m+1}\right|^2 \\
& + \frac{1}{2} \frac{1}{\Delta t} \left|x_u^{m+1}\right| \left|y^{m+1}\right|^2 \left|\tau^m - \tau_h^m\right|^2 + \frac{1}{2} \frac{1}{\Delta t} \left|x_u^{m+1}\right| \left|y^{m+1}\right|^2 \left|\tau_h^{m+1} - \tau_h^m\right|^2 \\
& - \frac{1}{2} \frac{1}{\Delta t} \left|x_u^{m+1}\right| \left|y^{m+1}\right|^2 \left|\tau^{m+1} - \tau^m\right|^2 + \frac{1}{\Delta t} \left|x_u^{m+1}\right| \left|y^{m+1}\right|^2 \left|\tau^{m+1} - \tau^m\right|^2.
\end{aligned}$$

We claim that the above sum is non-negative. To see this, we reorganize the terms in β. For clarity, we omit the factor $\frac{1}{2} \frac{1}{\Delta t} \left|x_u^{m+1}\right| \left|y^{m+1}\right|^2$ and preserve only the sign in front of the absolute values of the differences of the tangents. Thus, using the relation (3.21) we arrive at

$$\begin{aligned}
&\left|\tau^{m+1}-\tau_h^{m+1}\right|^2-\left|\tau^{m+1}-\tau_h^{m}\right|^2-\left|\tau^{m}-\tau_h^{m+1}\right|^2+\left|\tau^{m}-\tau_h^{m}\right|^2+\left|\tau_h^{m+1}-\tau_h^{m}\right|^2\\
&\quad+\left|\tau^{m+1}-\tau^{m}\right|^2\\
&=2-2\left(\tau^{m+1},\tau_h^{m+1}\right)-2+2\left(\tau^{m+1},\tau_h^{m}\right)-2+2\left(\tau^{m},\tau_h^{m+1}\right)+2-2\left(\tau^{m},\tau_h^{m}\right)\\
&\quad+\left|\tau_h^{m+1}-\tau_h^{m}\right|^2+\left|\tau^{m+1}-\tau^{m}\right|^2\\
&=\left|\tau^{m+1}-\tau^{m}\right|^2+\left|\tau_h^{m+1}-\tau_h^{m}\right|^2-2\left(\tau^{m+1}-\tau^{m},\tau_h^{m+1}-\tau_h^{m}\right)\\
&=\left|\left(\tau^{m+1}-\tau^{m}\right)-\left(\tau_h^{m+1}-\tau_h^{m}\right)\right|^2.
\end{aligned}$$

Therefore, the sum $\beta_1+\beta_2+\beta_3+\beta_4$ translates into

$$\beta=\frac{1}{2}\frac{1}{\Delta t}\left|y^{m+1}\right|^2\left|\left(\tau^{m+1}-\tau^{m}\right)-\left(\tau_h^{m+1}-\tau_h^{m}\right)\right|^2\left|x_u^{m+1}\right|.$$

Combining now β together with α given by (4.56) results in

$$\int_0^{2\pi}\alpha+\beta=\int_0^{2\pi}\frac{1}{4}\frac{1}{\Delta t}\left|y^{m+1}\right|^2\left|\left(\tau^{m+1}-\tau^{m}\right)-\left(\tau_h^{m+1}-\tau_h^{m}\right)\right|^2\left|x_u^{m+1}\right|\geq 0. \tag{4.75}$$

We note, that the above expression is on the left-hand side of (4.40). Since it is non-negative, it can be estimated by zero from below.
Collecting the results of this subsection and recalling (4.69) yields

$$\begin{aligned}
\int_0^{2\pi}B^m\geq&\frac{1}{\Delta t}\int_0^{2\pi}\Bigg(y_u^{m+1},\left(\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}-1\right)\left(\tau_h^{m+1}-\tau^{m+1}\right)\\
&\qquad\qquad+\frac{1}{2}\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}\left|\tau_h^{m+1}-\tau^{m+1}\right|^2\tau^{m+1}\Bigg)\\
&-\frac{1}{\Delta t}\int_0^{2\pi}\left(y_u^{m},\left(\frac{\left|x_{hu}^{m}\right|}{\left|x_u^{m}\right|}-1\right)\left(\tau_h^{m}-\tau^{m}\right)+\frac{1}{2}\frac{\left|x_{hu}^{m}\right|}{\left|x_u^{m}\right|}\left|\tau_h^{m}-\tau^{m}\right|^2\tau^{m}\right)\\
&-C\Bigg(\Delta t^2+\left\|\left|x_u^m\right|-\left|x_{hu}^m\right|\right\|^2+\left\|\tau^m-\tau_h^m\right\|^2+\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|^2\\
&\qquad+\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|^2+\left\|\tau^{m+1}-\tau_h^{m+1}\right\|^2+\frac{\Delta t}{h^2}\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2\Bigg).
\end{aligned} \tag{4.76}$$

If we choose ω, so that $C\omega^2\leq\frac{c_0}{8}$, the claim of the lemma follows by combining (4.40) with the results (4.57), (4.76) for A^m and B^m, respectively, with (4.75) and definition (4.39) of the function ζ^m.

□

As mentioned at the beginning of this chapter, several terms which appear on the right-hand side of (4.38) in the formulation of Lemma 4.6 have to be treated separately, since the discrete Gronwall argument cannot be applied directly. The following lemmas give us means to control such terms.

4.5 Tangent vector

The next Lemma 4.7 provides an estimate for the L^2-norm of the difference of the tangent vectors.

Lemma 4.7 (Tangent vector). *Suppose that bounds (4.16) hold. Then for* $m = 0, ..., M-1$ *and* $\varepsilon > 0$ *we have*

$$\begin{aligned}\left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2 \leq & \varepsilon \left(\left\|y^{m+1} - y_h^{m+1}\right\|^2 + \left\| |x_u^{m+1}| - |x_{hu}^{m+1}| \right\|^2 \right) \\ & + C_\varepsilon \left(h^2 + \left\|x^{m+1} - x_h^{m+1}\right\|^2 + h^{-1} RT^{m+1} \right),\end{aligned}$$

where RT^{m+1} *is given by (4.17).*

Proof. We evaluate the continuous equation (1.8) at $(m+1)\Delta t$ and subtract the discrete one (3.30). The result then is

$$\begin{aligned}\int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, \psi_{hu}\right) = & -\int_0^{2\pi} \left(y^{m+1}, \psi_h\right) \left|x_u^{m+1}\right| \\ & + \int_0^{2\pi} I_h \left[\left(y_h^{m+1}, \psi_h\right)\right] \left|x_{hu}^{m+1}\right| + \left\langle R_h^{m+1}, \psi_h \right\rangle .\end{aligned}$$

A simple calculation leads us to

$$\begin{aligned}\int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, \psi_{hu}\right) = & -\int_0^{2\pi} \left(y^{m+1}, \psi_h\right) \left(\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right) \\ & - \int_0^{2\pi} I_h \left[\left(I_h y^{m+1} - y_h^{m+1}, \psi_h\right)\right] \left|x_{hu}^{m+1}\right| \\ & + \int_0^{2\pi} I_h \left[\left(I_h y^{m+1}, \psi_h\right)\right] \left|x_{hu}^{m+1}\right| - \int_0^{2\pi} \left(I_h y^{m+1}, \psi_h\right) \left|x_{hu}^{m+1}\right| \\ & + \int_0^{2\pi} \left(I_h y^{m+1} - y^{m+1}, \psi_h\right) \left|x_{hu}^{m+1}\right| + \left\langle R_h^{m+1}, \psi_h \right\rangle .\end{aligned}$$

If we further apply (2.8), we arrive at

$$\begin{aligned}\int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, \psi_{hu}\right) = & -\int_0^{2\pi} \left(y^{m+1}, \psi_h\right) \left(\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right) \\ & - \int_0^{2\pi} I_h \left[\left(I_h y^{m+1} - y_h^{m+1}, \psi_h\right)\right] \left|x_{hu}^{m+1}\right| \\ & + \int_0^{2\pi} \left(I_h y^{m+1} - y^{m+1}, \psi_h\right) \left|x_{hu}^{m+1}\right| \\ & + \frac{1}{6} \sum_{j=1}^{N} h_j^2 \int_{I_j} \left(\left(I_h y^{m+1}\right)_u, \psi_{hu}\right) \left|x_{hu}^{m+1}\right| + \left\langle R_h^{m+1}, \psi_h \right\rangle .\end{aligned}$$

Inserting $\psi_h = I_h x^{m+1} - x_h^{m+1}$ as a test function into the equation above results in

$$\begin{aligned}
&\int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, x_u^{m+1} - x_{hu}^{m+1}\right) = -\int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, \left(I_h x^{m+1}\right)_u - x_u^{m+1}\right) \\
&\quad - \int_0^{2\pi} \left(y^{m+1}, I_h x^{m+1} - x_h^{m+1}\right)\left(\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right) \\
&\quad - \int_0^{2\pi} I_h\left[\left(I_h y^{m+1} - y_h^{m+1}, I_h x^{m+1} - x_h^{m+1}\right)\right]\left|x_{hu}^{m+1}\right| \\
&\quad + \int_0^{2\pi} \left(I_h y^{m+1} - y^{m+1}, I_h x^{m+1} - x_h^{m+1}\right)\left|x_{hu}^{m+1}\right| \\
&\quad + \frac{1}{6}\sum_{j=1}^{N} h_j^2 \int_{I_j} \left(\left(I_h y^{m+1}\right)_u, \left(I_h x^{m+1}\right)_u - x_{hu}^{m+1}\right)\left|x_{hu}^{m+1}\right| + \left\langle R_h^{m+1}, I_h x^{m+1} - x_h^{m+1}\right\rangle \\
&= \sum_{i=1}^{6} S_i.
\end{aligned}$$

We are going to examine these integrals separately. Using Cauchy-Schwarz and Young's inequalities along with an interpolation estimate we obtain for the first term

$$\begin{aligned}
S_1 = &-\int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, \left(I_h x^{m+1}\right)_u - x_u^{m+1}\right) \\
\leq &C\left\|\tau^{m+1} - \tau_h^{m+1}\right\|\left\|\left(I_h x^{m+1}\right)_u - x_u^{m+1}\right\| \leq \delta\left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2 + C_\delta h^2.
\end{aligned}$$

In a similar way we derive

$$\begin{aligned}
S_2 = &-\int_0^{2\pi} \left(y^{m+1}, I_h x^{m+1} - x_h^{m+1}\right)\left(\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right) \\
\leq &\frac{\varepsilon}{2}\left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 + C_\varepsilon\left(h^4 + \left\|x^{m+1} - x_h^{m+1}\right\|^2\right).
\end{aligned}$$

From (2.7), (4.16) and Young's inequality we deduce

$$\begin{aligned}
S_3 = &-\int_0^{2\pi} I_h\left[\left(I_h y^{m+1} - y_h^{m+1}, I_h x^{m+1} - x_h^{m+1}\right)\right]\left|x_{hu}^{m+1}\right| \\
\leq &\int_0^{2\pi} I_h\left[\varepsilon\left|I_h y^{m+1} - y_h^{m+1}\right|^2 + C_\varepsilon\left|I_h x^{m+1} - x_h^{m+1}\right|^2\right]\left|x_{hu}^{m+1}\right| \\
\leq &\varepsilon\left\|I_h y^{m+1} - y_h^{m+1}\right\|^2 + C_\varepsilon\left\|I_h x^{m+1} - x_h^{m+1}\right\|^2 \\
\leq &\varepsilon\left\|y^{m+1} - y_h^{m+1}\right\|^2 + C_\varepsilon\left(h^4 + \left\|x^{m+1} - x_h^{m+1}\right\|^2\right),
\end{aligned}$$

where the last estimate follows from (2.6) and triangle inequality.
An interpolation estimate and an upper bound on $\left|x_{hu}^{m+1}\right|$ from (4.16) give

$$S_4 = \int_0^{2\pi} \left(I_h y^{m+1} - y^{m+1}, I_h x^{m+1} - x_h^{m+1}\right)\left|x_{hu}^{m+1}\right| \leq C\left(h^4 + \left\|x^{m+1} - x_h^{m+1}\right\|^2\right).$$

Further, (4.16) and (4.27) together with the boundedness of the continuous solution imply

$$
\begin{aligned}
S_5 =& \frac{1}{6}\sum_{j=1}^{N} h_j^2 \int_{I_j} \left(\left(I_h y^{m+1}\right)_u, \left(I_h x^{m+1}\right)_u - x_{hu}^{m+1} \right) \left| x_{hu}^{m+1} \right| \\
\leq& Ch^2 \left(\int_0^{2\pi} \left| \left(I_h x^{m+1}\right)_u - x_u^{m+1} \right| + \int_0^{2\pi} \left| x_u^{m+1} - x_{hu}^{m+1} \right| \right) \\
\leq& Ch^2 \left(\int_0^{2\pi} \left| \left(I_h x^{m+1}\right)_u - x_u^{m+1} \right| + \int_0^{2\pi} \left| \left| x_u^{m+1} \right| - \left| x_{hu}^{m+1} \right| \right| + \int_0^{2\pi} \left| \tau^{m+1} - \tau_h^{m+1} \right| \right) \\
\leq& \frac{\varepsilon}{4} \left\| \left| x_u^{m+1} \right| - \left\| x_{hu}^{m+1} \right| \right\|^2 + \delta \left\| \tau^{m+1} - \tau_h^{m+1} \right\|^2 + \left(C_\varepsilon + C_\delta\right) h^3.
\end{aligned}
$$

Here, the inequality $\|f\|_{L^1(0,2\pi)} \leq \sqrt{2\pi}\, \|f\|_{L^2(0,2\pi)}$ was also used. The sixth term S_6 can be estimated in the following way. First, in view of the definition (3.27) of the remainder term we deduce

$$
\begin{aligned}
\left\langle R_h^{m+1}, \psi_h \right\rangle =& \sum_{k=0}^{m} \frac{1}{2} \int_0^{2\pi} \left(\tau_h^k, \psi_{hu}\right) \left| \tau_h^{k+1} - \tau_h^k \right|^2 \leq C \left\| \psi_{hu} \right\|_{L^\infty} \sum_{k=0}^{m} \left\| \tau_h^{k+1} - \tau_h^k \right\|^2 \\
\leq& Ch^{-\frac{1}{2}} \left\| \psi_{hu} \right\| \sum_{k=0}^{m} \left\| \tau_h^{k+1} - \tau_h^k \right\|^2 .
\end{aligned}
\tag{4.77}
$$

Inserting $\psi_h = I_h x^{m+1} - x_h^{m+1}$ into (4.77), using the relation (4.27) and recalling the notation (4.17) we arrive at

$$
\begin{aligned}
\left| S_6 \right| =& \left| \left\langle R_h^{m+1}, I_h x^{m+1} - x_h^{m+1} \right\rangle \right| \\
\leq& \varepsilon \left(h^2 + \left\| \left| x_u^{m+1} \right| - \left| x_{hu}^{m+1} \right| \right\|^2 \right) + \delta \left\| \tau^{m+1} - \tau_h^{m+1} \right\|^2 + \left(C_\varepsilon + C_\delta\right) h^{-1} RT^{m+1}.
\end{aligned}
$$

With the help of (3.21) we deduce

$$
\begin{aligned}
\left(\tau^{m+1} - \tau_h^{m+1}, x_u^{m+1} - x_{hu}^{m+1} \right) =& \left(\tau^{m+1} - \tau_h^{m+1}, \left| x_u^{m+1} \right| \tau^{m+1} - \left| x_{hu}^{m+1} \right| \tau_h^{m+1} \right) \\
=& \frac{1}{2} \left| \tau^{m+1} - \tau_h^{m+1} \right|^2 \left(\left| x_u^{m+1} \right| + \left| x_{hu}^{m+1} \right| \right).
\end{aligned}
$$

Finally, the above equality and bounds (4.1), (4.16) imply

$$
\begin{aligned}
\int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, x_u^{m+1} - x_{hu}^{m+1} \right) =& \frac{1}{2} \int_0^{2\pi} \left| \tau^{m+1} - \tau_h^{m+1} \right|^2 \left(\left| x_u^{m+1} \right| + \left| x_{hu}^{m+1} \right| \right) \\
\geq& C \left\| \tau^{m+1} - \tau_h^{m+1} \right\|^2 .
\end{aligned}
$$

Choosing δ small enough we complete the proof of this lemma.

□

4.6 Numerical scheme. Spatial derivative of the curvature vector

4.6.1 Fully discrete numerical scheme

Lemma 4.8 (Fully discrete scheme). *The fully discrete numerical scheme (1.11)-(1.12) to approximate the elastic flow of a curve can be written in the form*

$$\begin{aligned}\alpha_j^m \frac{x_j^{m+1}-x_j^m}{\Delta t} &+ \frac{1}{q_{j+1}^{m+1}} P_{j+1}^m \left(y_{j+1}^{m+1}-y_j^{m+1}\right) - \frac{1}{q_j^{m+1}} P_j^m \left(y_j^{m+1}-y_{j-1}^{m+1}\right) \\ &+ \frac{1}{4}\left(\left(\left|y_j^{m+1}\right|^2+\left|y_{j+1}^{m+1}\right|^2\right)\tau_{j+1}^{m+1} - \left(\left|y_{j-1}^{m+1}\right|^2+\left|y_j^{m+1}\right|^2\right)\tau_j^{m+1}\right) \\ &- \frac{\left(P_{j+1}^m\left(y_{j+1}^{m+1}-y_j^{m+1}\right),\tau_{j+1}^{m+1}\right)}{q_{j+1}^{m+1}}\tau_{j+1}^{m+1} + \frac{\left(P_j^m\left(y_j^{m+1}-y_{j-1}^{m+1}\right),\tau_j^{m+1}\right)}{q_j^{m+1}}\tau_j^{m+1} \\ &\qquad - \lambda\left(\tau_{j+1}^{m+1}-\tau_j^{m+1}\right) = 0,\end{aligned} \tag{4.78}$$

$$\alpha_j^{m+1} y_j^{m+1} - \left(\tau_{j+1}^{m+1}-\tau_j^{m+1}\right) + R_j^{m+1} = 0, \tag{4.79}$$

for $j = 1, ..., N$ periodically in N, where

$$\begin{aligned} &x_j^m = x_h^m(u_j),\ x_j^m = \left(x_{j,1}^m, ..., x_{j,n}^m\right), \\ &y_j^m = y_h^m(u_j),\ q_j^m = \left|x_j^m - x_{j-1}^m\right|,\ \alpha_j^m = \frac{1}{2}\left(q_j^m + q_{j+1}^m\right), \\ &\tau_j^m = \frac{x_j^m - x_{j-1}^m}{\left|x_j^m - x_{j-1}^m\right|},\ P_j^m = I - \tau_j^m \otimes \tau_j^m,\end{aligned} \tag{4.80}$$

$$R_j^{m+1} = -\frac{1}{2}\sum_{l=0}^{m}\left(\tau_{j+1}^l\left|\tau_{j+1}^{l+1}-\tau_{j+1}^l\right|^2 - \tau_j^l\left|\tau_j^{l+1}-\tau_j^l\right|^2\right). \tag{4.81}$$

Proof. First, we recall that equations (1.11)-(1.12) are equivalent to (3.29)-(3.30) (see, Corollary 3.8). In order to derive a finite difference scheme, we insert the following test functions $\phi_h, \psi_h = \varphi_j\, e^k$ $(k = 1, ..., n)$ into (3.29), (3.30), respectively, and obtain

$$\begin{aligned}&\int_0^{2\pi} I_h\left[\left(\frac{x_h^{m+1}-x_h^m}{\Delta t},\varphi_j\, e^k\right)\right]\left|x_{hu}^m\right| - \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1},\varphi_j'\, e^k\right)}{\left|x_{hu}^{m+1}\right|} \\ &-\frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^{m+1}\right|^2\right]\left(\tau_h^{m+1},\varphi_j'\, e^k\right) + \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1}-x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\varphi_j'\, e^k\right) \\ &+\lambda\int_0^{2\pi}\left(\tau_h^{m+1},\varphi_j'\, e^k\right) = I + II + III + IV + V = 0,\end{aligned} \tag{4.82}$$

$$\begin{aligned}&\int_0^{2\pi} I_h\left[\left(y_h^{m+1},\varphi_j\, e^k\right)\right]\left|x_{hu}^{m+1}\right| + \int_0^{2\pi}\left(\tau_h^{m+1},\varphi_j'\, e^k\right) + \left\langle R_h^{m+1},\varphi_j\, e^k\right\rangle \\ &\qquad = VI + VII + VIII = 0\end{aligned} \tag{4.83}$$

for $j = 1, ..., N$ periodically in N, $k = 1, ..., n$.

Below we present some properties of the basis functions φ_j, which will be helpful in finding the integrals listed above. We recall that due to the definition of the basis functions, the function φ_j, $j = 1, ..., N$ is different from zero only on $[u_{j-1}, u_{j+1}]$

$$\int_0^{2\pi} \varphi_j = \int_{u_{j-1}}^{u_j} \varphi_j + \int_{u_j}^{u_{j+1}} \varphi_j = \frac{h_j}{2} + \frac{h_{j+1}}{2},$$
$$\varphi_j'(u)|_{[u_{j-1},u_j]} = \frac{1}{h_j} \Rightarrow \int_{u_{j-1}}^{u_j} \varphi_j' = \frac{1}{h_j}\int_{u_{j-1}}^{u_j} 1 = \frac{1}{h_j} h_j = 1,$$
$$\varphi_j'(u)|_{[u_j,u_{j+1}]} = -\frac{1}{h_{j+1}} \Rightarrow \int_{u_j}^{u_{j+1}} \varphi_j' = -\frac{1}{h_{j+1}}\int_{u_j}^{u_{j+1}} 1 = -\frac{1}{h_{j+1}} h_{j+1} = -1,$$

where h_j is the length of a subinterval $I_j = [u_{j-1}, u_j]$, $j = 1, ..., N$ as defined in Section 2.2. Furthermore, for a piecewise linear function η_h the value $|\eta_{hu}|$ is constant on each subinterval I_j

$$|\eta_{hu}|_{I_j} = \frac{|\eta_h(u_j) - \eta_h(u_{j-1})|}{h_j} = \frac{|\eta_j - \eta_{j-1}|}{h_j}, \quad j = 1, ..., N.$$

Here we used the notations (4.80).
Recalling the definition of the Lagrange interpolation operator I_h we deduce

$$\begin{aligned}
I =& \int_0^{2\pi} I_h\left[\left(\frac{x_h^{m+1} - x_h^m}{\Delta t}, \varphi_j\, e^k\right)\right] |x_{hu}^m| \\
=& \int_0^{2\pi} \sum_{i=1}^{N} \left(\frac{x_h^{m+1}(u_i) - x_h^m(u_i)}{\Delta t}, \varphi_j(u_i)\, e^k\right) \varphi_i(u)\, |x_{hu}^m| \\
=& \int_{I_j \cup I_{j+1}} \left(\frac{x_h^{m+1}(u_j) - x_h^m(u_j)}{\Delta t}, e^k\right) \varphi_j(u)\, |x_{hu}^m| \\
=& \frac{\left|x_j^m - x_{j-1}^m\right|}{h_j} \frac{x_{j,k}^{m+1} - x_{j,k}^m}{\Delta t} \int_{u_{j-1}}^{u_j} \varphi_j(u) + \frac{\left|x_{j+1}^m - x_j^m\right|}{h_{j+1}} \frac{x_{j,k}^{m+1} - x_{j,k}^m}{\Delta t} \int_{u_j}^{u_{j+1}} \varphi_j(u) \\
=& \frac{1}{2}\left(q_j^m + q_{j+1}^m\right) \frac{x_{j,k}^{m+1} - x_{j,k}^m}{\Delta t} = \alpha_j^m \frac{x_{j,k}^{m+1} - x_{j,k}^m}{\Delta t}.
\end{aligned}$$

The second integral can be computed in the following way

$$\begin{aligned}
II =& -\int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, \varphi_j'\, e^k\right)}{\left|x_{hu}^{m+1}\right|} \\
=& -\frac{\left(P_j^m\left(y_j^{m+1} - y_{j-1}^{m+1}\right), e^k\right)}{\left|x_j^{m+1} - x_{j-1}^{m+1}\right|} \int_{u_{j-1}}^{u_j} \varphi_j' - \frac{\left(P_{j+1}^m\left(y_{j+1}^{m+1} - y_j^{m+1}\right), e^k\right)}{\left|x_{j+1}^{m+1} - x_j^{m+1}\right|} \int_{u_j}^{u_{j+1}} \varphi_j' \\
=& -\frac{\left[P_j^m\left(y_j^{m+1} - y_{j-1}^{m+1}\right)\right]_k}{\left|x_j^{m+1} - x_{j-1}^{m+1}\right|} \frac{1}{h_j} h_j - \frac{\left[P_{j+1}^m\left(y_{j+1}^{m+1} - y_j^{m+1}\right)\right]_k}{\left|x_{j+1}^{m+1} - x_j^{m+1}\right|} \left(-\frac{1}{h_{j+1}}\right) h_{j+1} \\
=& \frac{1}{q_{j+1}^{m+1}} \left[P_{j+1}^m\left(y_{j+1}^{m+1} - y_j^{m+1}\right)\right]_k - \frac{1}{q_j^{m+1}} \left[P_j^m\left(y_j^{m+1} - y_{j-1}^{m+1}\right)\right]_k,
\end{aligned}$$

where $\left[P_j^m y_j^{m+1}\right]_k$ denotes the k-th component of the vector $P_j^m y_j^{m+1}$.

Similarly as above, we derive

$$
\begin{aligned}
III =& -\frac{1}{2}\int_0^{2\pi} I_h\left[\left|y_h^{m+1}\right|^2\right]\left(\tau_h^{m+1},\varphi_j'\,e^k\right)\\
=& -\frac{1}{2}\int_0^{2\pi}\sum_{i=1}^{N}\left|y_h^{m+1}(u_i)\right|^2\varphi_i(u)\left(\tau_h^{m+1},\varphi_j'\,e^k\right)\\
=& -\frac{1}{2}\int_{I_j\cup I_{j+1}}\sum_{i=1}^{N}\left|y_h^{m+1}(u_i)\right|^2\varphi_i(u)\left(\tau_h^{m+1},\varphi_j'\,e^k\right)\\
=& -\frac{1}{2}\frac{1}{h_j}\tau_{j,k}^{m+1}\int_{u_{j-1}}^{u_j}\left(\left|y_{j-1}^{m+1}\right|^2\varphi_{j-1}(u)+\left|y_j^{m+1}\right|^2\varphi_j(u)\right)\\
&-\frac{1}{2}\left(-\frac{1}{h_{j+1}}\right)\tau_{j+1,k}^{m+1}\int_{u_j}^{u_{j+1}}\left(\left|y_j^{m+1}\right|^2\varphi_j(u)+\left|y_{j+1}^{m+1}\right|^2\varphi_{j+1}(u)\right)\\
=& -\frac{1}{2}\frac{1}{h_j}\tau_{j,k}^{m+1}\frac{h_j}{2}\left(\left|y_{j-1}^{m+1}\right|^2+\left|y_j^{m+1}\right|^2\right)+\frac{1}{2}\frac{1}{h_{j+1}}\tau_{j+1,k}^{m+1}\frac{h_{j+1}}{2}\left(\left|y_j^{m+1}\right|^2+\left|y_{j+1}^{m+1}\right|^2\right)\\
=&\frac{1}{4}\left(\left(\left|y_j^{m+1}\right|^2+\left|y_{j+1}^{m+1}\right|^2\right)\tau_{j+1,k}^{m+1}-\left(\left|y_{j-1}^{m+1}\right|^2+\left|y_j^{m+1}\right|^2\right)\tau_{j,k}^{m+1}\right).
\end{aligned}
$$

To continue, we note that the fourth integral in view of the definition of the tangent vector and relation $P_h^m\tau_h^m=0$ can be written as

$$
\int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1},x_{hu}^{m+1}-x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\varphi_j'\,e^k\right)=\int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1},\tau_h^{m+1}\right)}{\left|x_{hu}^{m+1}\right|}\left(\tau_h^{m+1},\varphi_j'\,e^k\right).
$$

Approaching the next integral in a similar way as II we receive

$$
\begin{aligned}
IV =& \int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1},\tau_h^{m+1}\right)}{\left|x_{hu}^{m+1}\right|}\left(\tau_h^{m+1},\varphi_j'\,e^k\right)\\
=&\frac{\left(P_j^m\left(y_j^{m+1}-y_{j-1}^{m+1}\right),\tau_j^{m+1}\right)}{\left|x_j^{m+1}-x_{j-1}^{m+1}\right|}\tau_{j,k}^{m+1}\int_{u_{j-1}}^{u_j}\varphi_j'\\
&+\frac{\left(P_{j+1}^m\left(y_{j+1}^{m+1}-y_j^{m+1}\right),\tau_{j+1}^{m+1}\right)}{\left|x_{j+1}^{m+1}-x_j^{m+1}\right|}\tau_{j+1,k}^{m+1}\int_{u_j}^{u_{j+1}}\varphi_j'\\
=& -\frac{\left(P_{j+1}^m\left(y_{j+1}^{m+1}-y_j^{m+1}\right),\tau_{j+1}^{m+1}\right)}{q_{j+1}^{m+1}}\tau_{j+1,k}^{m+1}+\frac{\left(P_j^m\left(y_j^{m+1}-y_{j-1}^{m+1}\right),\tau_j^{m+1}\right)}{q_j^{m+1}}\tau_{j,k}^{m+1}.
\end{aligned}
$$

Next, we treat the fifth integral in the same way as II and IV

$$
\begin{aligned}
V =&\lambda\int_0^{2\pi}\left(\tau_h^{m+1},\varphi_j'\,e^k\right)=\lambda\int_{u_{j-1}}^{u_j}\left(\tau_h^{m+1},\varphi_j'\,e^k\right)+\lambda\int_{u_j}^{u_{j+1}}\left(\tau_h^{m+1},\varphi_j'\,e^k\right)\\
=&\lambda\left(\tau_{j,k}^{m+1}\int_{u_{j-1}}^{u_j}\varphi_j'+\tau_{j+1,k}^{m+1}\int_{u_j}^{u_{j+1}}\varphi_j'\right)=-\lambda\left(\tau_{j+1,k}^{m+1}-\tau_{j,k}^{m+1}\right).
\end{aligned}
$$

A similar argument as for I can be used to obtain

$$\begin{aligned}
VI = &\int_0^{2\pi} I_h\left[\left(y_h^{m+1}, \varphi_j\, e^k\right)\right]\left|x_{hu}^{m+1}\right| = \int_0^{2\pi}\sum_{i=1}^{N}\left(y_h^{m+1}(u_i), \varphi_j(u_i)\, e^k\right)\varphi_i(u)\left|x_{hu}^{m+1}\right| \\
= &\int_0^{2\pi}\left(y_h^{m+1}(u_j), \varphi_j(u_j)\, e^k\right)\varphi_j(u)\left|x_{hu}^{m+1}\right| = \int_{I_j\cup I_{j+1}}\left(y_h^{m+1}(u_j), e^k\right)\varphi_j(u)\left|x_{hu}^{m+1}\right| \\
= &y_{j,k}^{m+1}\frac{\left|x_j^{m+1}-x_{j-1}^{m+1}\right|}{h_j}\int_{u_{j-1}}^{u_j}\varphi_j(u) + y_{j,k}^{m+1}\frac{\left|x_{j+1}^{m+1}-x_j^{m+1}\right|}{h_{j+1}}\int_{u_j}^{u_{j+1}}\varphi_j(u) \\
= &\frac{1}{2}\left(q_j^{m+1}+q_{j+1}^{m+1}\right)y_{j,k}^{m+1} = \alpha_j^{m+1}y_{j,k}^{m+1}.
\end{aligned}$$

In view of calculations for the fifth integral we have

$$VII = \int_0^{2\pi}\left(\tau_h^{m+1}, \varphi_j'\, e^k\right) = -\left(\tau_{j+1,k}^{m+1}-\tau_{j,k}^{m+1}\right).$$

Recalling the definition (3.27) of the remainder term we obtain for the last integral

$$VIII = \left\langle R_h^{m+1}, \varphi_j\, e^k\right\rangle = \frac{1}{2}\sum_{l=0}^{m}\int_0^{2\pi}\left(\tau_h^l, \varphi_j'\, e^k\right)\left|\tau_h^{l+1}-\tau_h^l\right|^2, \quad k = 1, ..., n.$$

Arguing in the same way we finally obtain

$$\begin{aligned}
\int_0^{2\pi}\left(\tau_h^l, \varphi_j'\, e^k\right)\left|\tau_h^{l+1}-\tau_h^l\right|^2 &= \int_{u_{j-1}}^{u_j}\left(\tau_h^l, \varphi_j'\, e^k\right)\left|\tau_h^{l+1}-\tau_h^l\right|^2 + \int_{u_j}^{u_{j+1}}\left(\tau_h^l, \varphi_j'\, e^k\right)\left|\tau_h^{l+1}-\tau_h^l\right|^2 \\
&= \left(\tau_{j,k}^l\left|\tau_j^{l+1}-\tau_j^l\right|^2\int_{u_{j-1}}^{u_j}\varphi_j' + \tau_{j+1,k}^l\left|\tau_{j+1}^{l+1}-\tau_{j+1}^l\right|^2\int_{u_j}^{u_{j+1}}\varphi_j'\right) \\
&= -\left(\tau_{j+1,k}^l\left|\tau_{j+1}^{l+1}-\tau_{j+1}^l\right|^2 - \tau_{j,k}^l\left|\tau_j^{l+1}-\tau_j^l\right|^2\right).
\end{aligned}$$

The result follows from the above calculations and recalling (4.81). □

Corollary 4.9. *For $j = 1, .., N$ and $m = 0, ..., M-1$ holds*

$$\left(\tau_{j+1}^{m+1}-\tau_j^{m+1}, \tau_j^{m+1}\right) = -\frac{1}{2}\left(\alpha_j^{m+1}\right)^2\left|y_j^{m+1}\right|^2 - RS_j^{m+1}, \tag{4.84}$$

$$\left(\tau_{j+1}^{m+1}, y_j^{m+1}\right) = \frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 + RM_j^{m+1}, \tag{4.85}$$

$$\left(\tau_j^{m+1}, y_j^{m+1}\right) = -\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 - RP_j^{m+1}, \tag{4.86}$$

where

$$RS_j^{m+1} = \alpha_j^{m+1}\left(y_j^{m+1}, R_j^{m+1}\right) + \frac{1}{2}\left|R_j^{m+1}\right|^2, \tag{4.87}$$

$$
\begin{aligned}
RM_j^{m+1} &= \left(y_j^{m+1}, R_j^{m+1}\right) + \frac{1}{2}\frac{1}{\alpha_j^{m+1}}\left|R_j^{m+1}\right|^2 - \frac{1}{\alpha_j^{m+1}}\left(\tau_{j+1}^{m+1}, R_j^{m+1}\right), \\
RP_j^{m+1} &= \left(y_j^{m+1}, R_j^{m+1}\right) + \frac{1}{2}\frac{1}{\alpha_j^{m+1}}\left|R_j^{m+1}\right|^2 + \frac{1}{\alpha_j^{m+1}}\left(\tau_j^{m+1}, R_j^{m+1}\right).
\end{aligned} \tag{4.88}
$$

Proof. From (3.21) and (4.79) we deduce

$$
\begin{aligned}
\left(\tau_{j+1}^{m+1} - \tau_j^{m+1}, \tau_j^{m+1}\right) &= -\frac{1}{2}\left|\tau_{j+1}^{m+1} - \tau_j^{m+1}\right|^2 = -\frac{1}{2}\left|\alpha_j^{m+1} y_j^{m+1} + R_j^{m+1}\right|^2 \\
&= -\frac{1}{2}\left(\alpha_j^{m+1}\right)^2\left|y_j^{m+1}\right|^2 - \alpha_j^{m+1}\left(y_j^{m+1}, R_j^{m+1}\right) - \frac{1}{2}\left|R_j^{m+1}\right|^2.
\end{aligned}
$$

The above calculation also implies

$$
\begin{aligned}
&\left(\tau_{j+1}^{m+1}, y_j^{m+1}\right) = \frac{1}{\alpha_j^{m+1}}\left(\tau_{j+1}^{m+1}, \left(\tau_{j+1}^{m+1} - \tau_j^{m+1}\right) - R_j^{m+1}\right) \\
&= \frac{1}{2}\frac{1}{\alpha_j^{m+1}}\left|\tau_{j+1}^{m+1} - \tau_j^{m+1}\right|^2 - \frac{1}{\alpha_j^{m+1}}\left(\tau_{j+1}^{m+1}, R_j^{m+1}\right) \\
&= \frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 + \left(y_j^{m+1}, R_j^{m+1}\right) + \frac{1}{2}\frac{1}{\alpha_j^{m+1}}\left|R_j^{m+1}\right|^2 - \frac{1}{\alpha_j^{m+1}}\left(\tau_{j+1}^{m+1}, R_j^{m+1}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left(\tau_j^{m+1}, y_j^{m+1}\right) = \frac{1}{\alpha_j^{m+1}}\left(\tau_j^{m+1}, \left(\tau_{j+1}^{m+1} - \tau_j^{m+1}\right) - R_j^{m+1}\right) \\
&= -\frac{1}{2}\frac{1}{\alpha_j^{m+1}}\left|\tau_{j+1}^{m+1} - \tau_j^{m+1}\right|^2 - \frac{1}{\alpha_j^{m+1}}\left(\tau_j^{m+1}, R_j^{m+1}\right) \\
&= -\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 - \left(y_j^{m+1}, R_j^{m+1}\right) - \frac{1}{2}\frac{1}{\alpha_j^{m+1}}\left|R_j^{m+1}\right|^2 - \frac{1}{\alpha_j^{m+1}}\left(\tau_j^{m+1}, R_j^{m+1}\right).
\end{aligned}
$$

Together with the following abbreviations (4.87)-(4.88) the assertion follows.

□

4.6.2 Spatial derivative of the curvature vector

The aim of this section is to prove an error bound for the space derivative of the curvature vector. In the next lemma we shall make use of the results of Section 4.6.1.

Lemma 4.10 (Spatial derivative of the curvature vector)**.** *Suppose (4.16) holds. Then there exist $h_0 > 0$ and $0 < \omega \le 1$, such that for all $h \le h_0$ and $\Delta t \le \omega h^3$ we have for $\varepsilon > 0$ and $m = 0, ..., M-1$*

$$
\begin{aligned}
\left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 \le \left(\varepsilon + C\Delta t^2 h^{-4}\right)\left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + Ch^{-5} RT^{m+1} \\
+ C_\varepsilon\Big(h^2 + \Delta t^2 + \|x^m - x_h^m\|^2 + \|y^m - y_h^m\|^2 + \||x_u^m| - |x_{hu}^m|\|^2 \\
+ \left\|x^{m+1} - x_h^{m+1}\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2 + \left\||x_u^{m+1}| - |x_{hu}^{m+1}|\right\|^2\Big),
\end{aligned}
$$

where RT^{m+1} is given by (4.17). The constants are independent of h and Δt.

Proof. Recalling the definition of the projection matrix, we split the difference $y_u^{m+1} - y_{hu}^{m+1}$ in the following way

$$y_u^{m+1} - y_{hu}^{m+1} = P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right) + \left(y_u^{m+1} - y_{hu}^{m+1}, \tau_h^m\right) \tau_h^m. \tag{4.89}$$

We square (4.89), then integrate the result over $[0, 2\pi]$ and using Young's inequality get

$$\int_0^{2\pi} \left|y_u^{m+1} - y_{hu}^{m+1}\right|^2 \leq 2 \int_0^{2\pi} \left|P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right)\right|^2 + 2 \int_0^{2\pi} \left(y_u^{m+1} - y_{hu}^{m+1}, \tau_h^m\right)^2 .$$

In order to obtain the first term on the right-hand side of the above equation, we take the difference between equations (4.13) and (1.7), where the latter has been evaluated at a discrete test function ϕ_h and $(m+1)\Delta t$. This results in

$$\begin{aligned}
&\int_0^{2\pi} I_h \left[\left(\frac{x_h^{m+1} - x_h^m}{\Delta t}, \phi_h\right)\right] |x_{hu}^m| - \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, \phi_{hu}\right)}{\left|x_{hu}^{m+1}\right|} \\
&- \frac{1}{2} \int_0^{2\pi} I_h \left[\left|y_h^{m+1}\right|^2\right] \left(\tau_h^{m+1}, \phi_{hu}\right) + \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1} - x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1}, \phi_{hu}\right) \\
&\qquad + \lambda \int_0^{2\pi} \left(\tau_h^{m+1}, \phi_{hu}\right) \\
&- \left(\int_0^{2\pi} \left(x_t^{m+1}, \phi_h\right) \left|x_u^{m+1}\right| - \int_0^{2\pi} \frac{1}{\left|x_u^{m+1}\right|} \left(P^{m+1} y_u^{m+1}, \phi_{hu}\right) \right. \\
&\qquad \left. - \frac{1}{2} \int_0^{2\pi} \left|y^{m+1}\right|^2 \left(\tau^{m+1}, \phi_{hu}\right) + \lambda \int_0^{2\pi} \left(\tau^{m+1}, \phi_{hu}\right)\right) = 0.
\end{aligned}$$

After rearranging the terms the above equation reads as follows

$$\begin{aligned}
&\int_0^{2\pi} \frac{1}{\left|x_{hu}^{m+1}\right|} \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right), \phi_{hu}\right) \\
&= - \int_0^{2\pi} \left(\frac{1}{\left|x_u^{m+1}\right|} P^{m+1} y_u^{m+1} - \frac{1}{\left|x_{hu}^{m+1}\right|} P_h^m y_u^{m+1}, \phi_{hu}\right) \\
&+ \int_0^{2\pi} \left(x_t^{m+1}, \phi_h\right) \left|x_u^{m+1}\right| - \int_0^{2\pi} I_h \left[\left(\frac{x_h^{m+1} - x_h^m}{\Delta t}, \phi_h\right)\right] |x_{hu}^m| \\
&+ \frac{1}{2} \int_0^{2\pi} I_h \left[\left|y_h^{m+1}\right|^2\right] \left(\tau_h^{m+1}, \phi_{hu}\right) - \frac{1}{2} \int_0^{2\pi} \left|y^{m+1}\right|^2 \left(\tau^{m+1}, \phi_{hu}\right) \\
&- \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1} - x_{hu}^m\right)}{\left|x_{hu}^{m+1}\right|^3} \left(x_{hu}^{m+1}, \phi_{hu}\right) + \lambda \int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, \phi_{hu}\right),
\end{aligned} \tag{4.90}$$

where the last term on the right-hand side in view of (4.20) and (4.21) can be expressed in the following way

$$\begin{aligned}
\lambda \int_0^{2\pi} \left(\tau^{m+1} - \tau_h^{m+1}, \phi_{hu}\right) = &\lambda \int_0^{2\pi} I_h \left[\left(y_h^{m+1}, \phi_h\right)\right] \left|x_{hu}^{m+1}\right| - \lambda \int_0^{2\pi} \left(y^{m+1}, \phi_h\right) \left|x_u^{m+1}\right| \\
&+ \lambda \left\langle R_h^{m+1}, \phi_h\right\rangle .
\end{aligned}$$

Inserting $\phi_h = I_h y^{m+1} - y_h^{m+1}$ as a test function into (4.90) and taking into account the above equality we arrive at

$$\begin{aligned}
&\int_0^{2\pi} \frac{1}{|x_{hu}^{m+1}|} \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right), \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right) \\
&= - \int_0^{2\pi} \left(\frac{1}{|x_u^{m+1}|} P^{m+1} y_u^{m+1} - \frac{1}{|x_{hu}^{m+1}|} P_h^m y_u^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1} \right) \\
&\quad + \left\{ \int_0^{2\pi} \left(x_t^{m+1}, I_h y^{m+1} - y_h^{m+1}\right) |x_u^{m+1}| \right. \\
&\quad\quad \left. - \int_0^{2\pi} I_h \left[\left(\frac{x_h^{m+1} - x_h^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1} \right) \right] |x_{hu}^m| \right\} \\
&\quad + \frac{1}{2} \left\{ \int_0^{2\pi} I_h \left[\left|y_h^{m+1}\right|^2 \right] \left(\tau_h^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right) \right. \\
&\quad\quad \left. - \int_0^{2\pi} \left|y^{m+1}\right|^2 \left(\tau^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right) \right\} \qquad (4.91) \\
&\quad - \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1}, x_{hu}^{m+1} - x_{hu}^m\right)}{|x_{hu}^{m+1}|^3} \left(x_{hu}^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right) \\
&\quad + \lambda \left\{ \int_0^{2\pi} I_h \left[\left(y_h^{m+1}, I_h y^{m+1} - y_h^{m+1}\right) \right] |x_{hu}^{m+1}| \right. \\
&\quad\quad \left. - \int_0^{2\pi} \left(y^{m+1}, I_h y^{m+1} - y_h^{m+1}\right) |x_u^{m+1}| \right\} + \lambda \left\langle R_h^{m+1}, I_h y^{m+1} - y_h^{m+1} \right\rangle \\
&= \sum_{i=1}^{6} S_i.
\end{aligned}$$

Let us estimate the terms on the right-hand side of (4.91). To begin, we split the first interval into three

$$\begin{aligned}
S_1 &= - \int_0^{2\pi} \left(\frac{1}{|x_u^{m+1}|} P^{m+1} y_u^{m+1} - \frac{1}{|x_{hu}^{m+1}|} P_h^m y_u^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1} \right) \\
&= - \int_0^{2\pi} \left(\frac{1}{|x_u^{m+1}|} \left(P^{m+1} - P^m\right) y_u^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1} \right) \\
&\quad - \int_0^{2\pi} \left(\left(\frac{1}{|x_u^{m+1}|} - \frac{1}{|x_{hu}^{m+1}|} \right) P^m y_u^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1} \right) \\
&\quad - \int_0^{2\pi} \left(\frac{1}{|x_{hu}^{m+1}|} \left(P^m - P_h^m\right) y_u^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1} \right).
\end{aligned}$$

Next, we use (4.26), (4.63) to rewrite the differences $P^m - P_h^m$, $P^{m+1} - P^m$, respectively. Then, employing Cauchy-Schwarz and Young's inequalities, an interpolation estimate and bounds (4.16) we infer

$$
\begin{aligned}
S_1 \leq & C \int_0^{2\pi} \left(\left|\tau^{m+1} - \tau^m\right| + \left|\tau^m - \tau_h^m\right| + \left|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right|\right) \left|\left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right| \\
\leq & \delta \left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 + C_\delta \left(h^2 + \Delta t^2 + \left\|\tau^m - \tau_h^m\right\|^2 + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2\right).
\end{aligned}
$$

It will be convenient to write S_2 as a sum of three terms

$$
\begin{aligned}
S_2 = & \int_0^{2\pi} \left(x_t^{m+1}, I_h y^{m+1} - y_h^{m+1}\right) \left(\left|x_u^{m+1}\right| - \left|x_{hu}^m\right|\right) \\
& + \int_0^{2\pi} \left(x_t^{m+1} - \frac{x_h^{m+1} - x_h^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right) \left|x_{hu}^m\right| \\
& - \left\{\int_0^{2\pi} \left(I_h \left[\left(\frac{x_h^{m+1} - x_h^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right)\right]\right) \left|x_{hu}^m\right| \right. \\
& \quad \left. - \int_0^{2\pi} \left(\frac{x_h^{m+1} - x_h^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right) \left|x_{hu}^m\right| \right\} \\
= & S_{2,1} + S_{2,2} + S_{2,3}.
\end{aligned}
$$

Let us consider these expressions in more detail. Thus, the reverse triangle inequality and an interpolation estimate along with the boundedness of the continuous solution imply

$$
\begin{aligned}
S_{2,1} = & \int_0^{2\pi} \left(x_t^{m+1}, I_h y^{m+1} - y_h^{m+1}\right) \left(\left|x_u^{m+1}\right| - \left|x_{hu}^m\right|\right) \\
\leq & C \left(h^4 + \Delta t^2 + \left\|\left|x_u^m\right| - \left|x_{hu}^m\right|\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2\right).
\end{aligned}
$$

Recalling the definition of the error we may write

$$
\begin{aligned}
S_{2,2} = & \int_0^{2\pi} \left(x_t^{m+1} - I_h x_t^{m+1}, I_h y^{m+1} - y_h^{m+1}\right) \left|x_{hu}^m\right| \\
& + \int_0^{2\pi} \left(\frac{e_h^{m+1} - e_h^m}{\Delta t}, I_h y^{m+1} - y_h^{m+1}\right) \left|x_{hu}^m\right| \\
& + \int_0^{2\pi} \left(I_h \left[x_t^{m+1} - \frac{x^{m+1} - x^m}{\Delta t}\right], I_h y^{m+1} - y_h^{m+1}\right) \left|x_{hu}^m\right|.
\end{aligned}
$$

Exploiting next the Taylor expansion, an interpolation estimate, Cauchy-Schwarz and Young's inequalities results in

$$
S_{2,2} \leq \varepsilon \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C_\varepsilon \left(h^4 + \Delta t^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2\right).
$$

Next, we use (2.8) and Cauchy-Schwarz inequality and find that

$$
\begin{aligned}
S_{2,3} = & -\frac{1}{6} \sum_{j=1}^N h_j^2 \int_{I_j} \left(\frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right) \left|x_{hu}^m\right| \\
\leq & C h^2 \left\|\frac{x_{hu}^{m+1} - x_{hu}^m}{\Delta t}\right\| \left\|\left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right\|.
\end{aligned}
$$

From the error decomposition, inverse estimate (2.11), boundedness of the continuous solution, Young's inequality and an interpolation estimate we deduce

$$\begin{aligned}
S_{2,3} \leq & Ch^2 \left(\left\| \frac{e_{hu}^{m+1} - e_{hu}^m}{\Delta t} \right\| + \left\| \left(I_h \left[\frac{x^{m+1} - x^m}{\Delta t} \right] \right)_u \right\| \right) \left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| \\
\leq & C \left(\left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\| + h \right) \left\| I_h y^{m+1} - y_h^{m+1} \right\| \\
\leq & \varepsilon \left\| \frac{e_h^{m+1} - e_h^m}{\Delta t} \right\|^2 + C_\varepsilon \left(h^2 + \left\| y^{m+1} - y_h^{m+1} \right\|^2 \right).
\end{aligned}$$

Let us now consider S_3. In order to apply (2.8), we first add a zero term and organize the resulting expressions in the following way

$$\begin{aligned}
S_3 = & \frac{1}{2} \int_0^{2\pi} I_h \left[\left| y_h^{m+1} \right|^2 \right] \left(\tau_h^{m+1}, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) \\
& - \frac{1}{2} \int_0^{2\pi} \left| y^{m+1} \right|^2 \left(\tau^{m+1}, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) \\
= & \frac{1}{2} \int_0^{2\pi} \left(I_h \left[\left| y_h^{m+1} \right|^2 \right] - \left| y_h^{m+1} \right|^2 \right) \left(\tau_h^{m+1}, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) \\
& + \frac{1}{2} \int_0^{2\pi} \left(\left| y_h^{m+1} \right|^2 \tau_h^{m+1} - \left| y^{m+1} \right|^2 \tau^{m+1}, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) \\
= & \frac{1}{2} \frac{1}{6} \sum_{j=1}^N h_j^2 \int_{I_j} \left| y_{hu}^{m+1} \right|^2 \left(\tau_h^{m+1}, \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) \\
& + \frac{1}{2} \int_0^{2\pi} \left(\left(\left| y_h^{m+1} \right|^2 - \left| y^{m+1} \right|^2 \right) \tau_h^{m+1} - \left| y^{m+1} \right|^2 \left(\tau^{m+1} - \tau_h^{m+1} \right), \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right) \\
= & S_{3,1} + S_{3,2}.
\end{aligned}$$

Further, we treat these integrals separately. Taking out of the integral the L^∞-norm of $\left| y_{hu}^{m+1} \right|$ and using afterwards the inverse (2.11) and Young's inequalities results in

$$\begin{aligned}
S_{3,1} \leq & Ch^2 \left\| y_{hu}^{m+1} \right\|_{L_\infty} \left\| y_{hu}^{m+1} \right\| \left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| \\
\leq & Ch \left\| y_h^{m+1} \right\|_{L_\infty} \left(\left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| + \left\| \left(I_h y^{m+1} \right)_u \right\| \right) \left\| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right\| \\
\leq & C \left(\left\| I_h y^{m+1} - y_h^{m+1} \right\| + h \right) \left(\left\| \left(I_h y^{m+1} \right)_u - y_u^{m+1} \right\| + \left\| y_u^{m+1} - y_{hu}^{m+1} \right\| \right) \\
\leq & \delta \left\| y_u^{m+1} - y_{hu}^{m+1} \right\|^2 + C_\delta \left(h^2 + \left\| y^{m+1} - y_h^{m+1} \right\|^2 \right).
\end{aligned}$$

To estimate $S_{3,2}$, we employ Cauchy-Schwarz and Young's inequalities as well as (4.16)

$$\begin{aligned}
S_{3,2} \leq & C \int_0^{2\pi} \left(\left| y^{m+1} - y_h^{m+1} \right| \left(\left| y^{m+1} \right| + \left| y_h^{m+1} \right| \right) + \left| \tau^{m+1} - \tau_h^{m+1} \right| \right) \left| \left(I_h y^{m+1} \right)_u - y_{hu}^{m+1} \right| \\
\leq & C \left(\left\| y^{m+1} - y_h^{m+1} \right\| + \left\| \tau^{m+1} - \tau_h^{m+1} \right\| \right) \left(\left\| \left(I_h y^{m+1} \right)_u - y_u^{m+1} \right\| + \left\| y_u^{m+1} - y_{hu}^{m+1} \right\| \right) \\
\leq & \delta \left\| y_u^{m+1} - y_{hu}^{m+1} \right\|^2 + C_\delta \left(h^2 + \left\| y^{m+1} - y_h^{m+1} \right\|^2 + \left\| \tau^{m+1} - \tau_h^{m+1} \right\|^2 \right).
\end{aligned}$$

We split the next term into a sum as it was done for S_9 in Lemma 4.4

$$
\begin{aligned}
S_4 =& \int_0^{2\pi} \left(P^m y_u^{m+1}, \left(I_h\left[x^{m+1}-x^m\right]\right)_u\right)\left(x_u^{m+1}, \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right)\left(\frac{1}{\left|x_u^{m+1}\right|^3}-\frac{1}{\left|x_{hu}^{m+1}\right|^3}\right)\\
&+\int_0^{2\pi}\frac{\left(\left(P^m-P_h^m\right)y_u^{m+1},\left(I_h\left[x^{m+1}-x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_u^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
&+\int_0^{2\pi}\frac{\left(P_h^m y_u^{m+1},\left(I_h\left[x^{m+1}-x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_u^{m+1}-x_{hu}^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
&+\int_0^{2\pi}\frac{\left(P_h^m\left(y_u^{m+1}-y_{hu}^{m+1}\right),\left(I_h\left[x^{m+1}-x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
&+\int_0^{2\pi}\frac{\left(P_h^m y_{hu}^{m+1},\left(I_h\left[x^{m+1}-x^m\right]\right)_u-\left(x_{hu}^{m+1}-x_{hu}^m\right)\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
&-\int_0^{2\pi}\frac{\left(P^m y_u^{m+1},\left(I_h\left[x^{m+1}-x^m\right]\right)_u\right)}{\left|x_u^{m+1}\right|^3}\left(x_u^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
=&\sum_{i=1}^{6} S_{4,i}.
\end{aligned}
$$

The following estimates for S_4 use the boundedness of the continuous solution. An interpolation estimate and (4.16), on the other hand, provide

$$
\begin{aligned}
S_{4,1} \leq& C\Delta t\int_0^{2\pi}\left|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right|\left|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right|\\
\leq& C\Delta t\left\|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right\|\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|\\
\leq&\Delta t^2\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|^2+C\left(h^4+\Delta t^4+\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|^2\right).
\end{aligned}
$$

Arguing as above and exploiting in addition (4.26) for the difference $P^m - P_h^m$ we obtain

$$
\begin{aligned}
S_{4,2} \leq& C\Delta t\int_0^{2\pi}\left|\tau^m-\tau_h^m\right|\left|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right| \leq C\Delta t\left\|\tau^m-\tau_h^m\right\|\left\|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right\|\\
\leq&\Delta t^2\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|^2+C\left(h^4+\Delta t^4+\left\|\tau^m-\tau_h^m\right\|^2\right).
\end{aligned}
$$

Next, from (4.16) and (4.27) we infer

$$
\begin{aligned}
S_{4,3} =& \int_0^{2\pi}\frac{\left(P_h^m y_u^{m+1},\left(I_h\left[x^{m+1}-x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_u^{m+1}-x_{hu}^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
\leq& C\Delta t\int_0^{2\pi}\left(\left|\tau^{m+1}-\tau_h^{m+1}\right|+\left|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right|\right)\left|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right|\\
\leq& C\Delta t\left(\left\|\tau^{m+1}-\tau_h^{m+1}\right\|+\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|\right)\left(h+\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|\right)\\
\leq&\Delta t^2\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|^2+C\left(h^4+\Delta t^4+\left\|\tau^{m+1}-\tau_h^{m+1}\right\|^2+\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|^2\right).
\end{aligned}
$$

Analogously, we get

$$\begin{aligned}
S_{4,4} =& \int_0^{2\pi} \frac{\left(P_h^m\left(y_u^{m+1}-y_{hu}^{m+1}\right),\left(I_h\left[x^{m+1}-x^m\right]\right)_u\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
\leq& C\Delta t\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|\left\|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right\|\\
\leq& \left(\delta+C\Delta t\right)\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|^2+C_\delta\left(h^4+\Delta t^4\right).
\end{aligned}$$

Recalling the definition of the error, using (4.16) and (2.11) as well as Young's inequality we receive

$$\begin{aligned}
S_{4,5} =& \int_0^{2\pi} \frac{\left(P_h^m y_{hu}^{m+1},\left(I_h\left[x^{m+1}-x^m\right]\right)_u-\left(x_{hu}^{m+1}-x_{hu}^m\right)\right)}{\left|x_{hu}^{m+1}\right|^3}\left(x_{hu}^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
\leq& C\int_0^{2\pi}\left|y_{hu}^{m+1}\right|\left|e_{hu}^{m+1}-e_{hu}^m\right|\left|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right|\\
\leq& Ch^{-1}\left\|y_h^{m+1}\right\|_{L^\infty}\left\|e_{hu}^{m+1}-e_{hu}^m\right\|\left\|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right\|\\
\leq& C\Delta t h^{-2}\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|\left(h+\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|\right)\\
\leq& \delta\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|^2+C_\delta\left(h^2+\Delta t^2 h^{-4}\left\|\frac{e_h^{m+1}-e_h^m}{\Delta t}\right\|^2\right).
\end{aligned}$$

From the boundedness of the continuous solution and (4.16) we infer

$$\begin{aligned}
S_{4,6} =& -\int_0^{2\pi}\frac{\left(P^m y_u^{m+1},\left(I_h\left[x^{m+1}-x^m\right]\right)_u\right)}{\left|x_u^{m+1}\right|^3}\left(x_u^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\\
\leq& C\Delta t\left\|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right\|\leq\delta\left\|y_u^{m+1}-y_{hu}^{m+1}\right\|^2+C_\delta\left(h^2+\Delta t^2\right).
\end{aligned}$$

We observe that the fifth term can be written as

$$\begin{aligned}
S_5 =& \lambda\left(\int_0^{2\pi} I_h\left[\left(y_h^{m+1},I_h y^{m+1}-y_h^{m+1}\right)\right]\left|x_{hu}^{m+1}\right|-\int_0^{2\pi}\left(y_h^{m+1},I_h y^{m+1}-y_h^{m+1}\right)\left|x_{hu}^{m+1}\right|\right)\\
&-\lambda\int_0^{2\pi}\left(y^{m+1}-y_h^{m+1},I_h y^{m+1}-y_h^{m+1}\right)\left|x_{hu}^{m+1}\right|\\
&-\lambda\int_0^{2\pi}\left(y^{m+1},I_h y^{m+1}-y_h^{m+1}\right)\left(\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right)\\
=& S_{5,1}+S_{5,2}+S_{5,3}.
\end{aligned}$$

By (2.8) we have

$$\begin{aligned}
S_{5,1} =& \frac{\lambda}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left(y_{hu}^{m+1},\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\left|x_{hu}^{m+1}\right|\\
=& \frac{\lambda}{6}\sum_{j=1}^N h_j^2\int_{I_j}\left(-\left|\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right|^2+\left(\left(I_h y^{m+1}\right)_u,\left(I_h y^{m+1}\right)_u-y_{hu}^{m+1}\right)\right)\left|x_{hu}^{m+1}\right|.
\end{aligned}$$

Combining (2.6) with Cauchy-Schwarz inequality we find that

$$\begin{aligned} S_{5,1} \leq & Ch^2 \int_0^{2\pi} \left|\left(I_h y^{m+1}\right)_u\right| \left|\left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right| \left|x_{hu}^{m+1}\right| \\ \leq & Ch^2 \left\|\left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right\| \leq \delta \left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 + C_\delta h^3. \end{aligned}$$

Analogously we obtain for the sum of two integrals

$$\begin{aligned} S_{5,2} + S_{5,3} \leq & C \left(\left\|y^{m+1} - y_h^{m+1}\right\| + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|\right) \left\|I_h y^{m+1} - y_h^{m+1}\right\| \\ \leq & C \left(h^4 + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2\right). \end{aligned}$$

The last term to be estimated on the right-hand side of (4.91) is S_6. Hence, inserting the test function $\psi_h = I_h y^{m+1} - y_h^{m+1}$ into (4.77) we obtain in view of (4.17)

$$|S_6| = \left|\lambda \left\langle R_h^{m+1}, I_h y^{m+1} - y_h^{m+1}\right\rangle\right| \leq \delta \left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 + C_\delta h^2 + C_\delta h^{-1} RT^{m+1}.$$

Let us finally rewrite the left-hand side of (4.91). Thus, from the definition of the projection matrix $P_h^m = I - \tau_h^m \otimes \tau_h^m$, from which the relation $P_h^m \tau_h^m = 0$ and the symmetry property follow, we infer after adding a zero term

$$\begin{aligned} & \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right), \left(I_h y^{m+1}\right)_u - y_{hu}^{m+1}\right) \\ & = \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right), y_u^{m+1} - y_{hu}^{m+1}\right) + \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right), \left(I_h y^{m+1}\right)_u - y_u^{m+1}\right) \\ & = \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right), y_u^{m+1} - y_{hu}^{m+1} - \tau_h^m \left(y_u^{m+1} - y_{hu}^{m+1}, \tau_h^m\right)\right) \\ & \quad + \left(\left(y_u^{m+1} - y_{hu}^{m+1}\right), P_h^m \tau_h^m \left(y_u^{m+1} - y_{hu}^{m+1}, \tau_h^m\right)\right) \\ & \quad + \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right), \left(I_h y^{m+1}\right)_u - y_u^{m+1}\right) \\ & = \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right)\right)^2 + \left(P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right), \left(I_h y^{m+1}\right)_u - y_u^{m+1}\right). \end{aligned}$$

Next, we estimate the left-hand side of (4.91) from below using an upper bound $\left|x_{hu}^{m+1}\right| \leq 4C_0$. Further, combination of estimates for $S_1, .., S_6$ and the relation above imply

$$\begin{aligned} & \frac{1}{4C_0} \left\|P_h^m \left(y_u^{m+1} - y_{hu}^{m+1}\right)\right\|^2 \leq \left(8\delta + C\Delta t + 3\Delta t^2\right) \left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 \\ & + \left(\varepsilon + C_\delta \Delta t^2 h^{-4}\right) \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C_\delta h^{-1} RT^{m+1} \\ & + \left(C_\delta + C_\varepsilon\right)\left(h^2 + \Delta t^2\right) + C_\delta \left(\left\|\tau^m - \tau_h^m\right\|^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2\right) \\ & + \left(C_\delta + C_\varepsilon\right) \left(\left\|\left|x_u^m\right| - \left|x_{hu}^m\right|\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2 + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2\right) \end{aligned} \tag{4.92}$$

The next step consists in deriving a suitable form for the second summand on the right-hand side of (4.89). In the following calculations the relation (4.70) will be fundamental. Therefore, we repeat it here for convenience at the $(m+1)$-st time level

$$\left(y_u^{m+1}, \tau^{m+1}\right) = \left(y^{m+1}, \tau^{m+1}\right)_u - \left(y^{m+1}, \tau_u^{m+1}\right) = -\left|x_u^{m+1}\right| \left|y^{m+1}\right|^2. \tag{4.93}$$

In order to derive the discrete equivalent of (4.93), we use (4.85) and (4.86)

$$\begin{aligned}\left(y_j^{m+1}-y_{j-1}^{m+1},\tau_j^{m+1}\right)&=\left(y_j^{m+1},\tau_j^{m+1}\right)-\left(y_{j-1}^{m+1},\tau_j^{m+1}\right)\\&=-\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2-\frac{1}{2}\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2-RP_j^{m+1}-RM_{j-1}^{m+1},\end{aligned}\tag{4.94}$$

where RP_j^{m+1}, RM_j^{m+1} given by (4.88).
Combining (4.93) with (4.94) we find in I_j

$$\begin{aligned}\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^{m}\right)&=\left(y_u^{m+1},\tau_j^{m+1}\right)-\left(y_{hu}^{m+1},\tau_j^{m+1}\right)+\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^{m}-\tau_j^{m+1}\right)\\&=\left(y_u^{m+1},\tau^{m+1}\right)-\left(\frac{y_j^{m+1}-y_{j-1}^{m+1}}{h_j},\tau_j^{m+1}\right)\\&\quad+\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^{m}-\tau_j^{m+1}\right)+\left(y_u^{m+1},\tau_j^{m+1}-\tau^{m+1}\right)\\&=-\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2+\left(y_u^{m+1},\tau_j^{m+1}-\tau^{m+1}\right)\\&\quad+\frac{1}{2}\frac{\alpha_j^{m+1}}{h_j}\left|y_j^{m+1}\right|^2+\frac{1}{2}\frac{\alpha_{j-1}^{m+1}}{h_j}\left|y_{j-1}^{m+1}\right|^2+\frac{1}{h_j}\left(RP_j^{m+1}+RM_{j-1}^{m+1}\right)\\&\quad+\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^{m}-\tau_j^{m+1}\right).\end{aligned}$$

Further, the definition $\alpha_j^{m+1}=\dfrac{1}{2}\left(q_j^{m+1}+q_{j+1}^{m+1}\right)$ allows us to write

$$\begin{aligned}\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^{m}\right)&=-\left|x_u^{m+1}\right|\left|y^{m+1}\right|^2+\left(y_u^{m+1},\tau_j^{m+1}-\tau^{m+1}\right)\\&\quad+\frac{q_j^{m+1}+q_{j+1}^{m+1}}{4h_j}\left|y_j^{m+1}\right|^2+\frac{q_{j-1}^{m+1}+q_j^{m+1}}{4h_j}\left|y_{j-1}^{m+1}\right|^2\\&\quad+\frac{1}{h_j}\left(RP_j^{m+1}+RM_{j-1}^{m+1}\right)+\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^{m}-\tau_j^{m+1}\right).\end{aligned}$$

Noting that $\left|x_{hu}^{m+1}\right|\big|_{I_j}=\dfrac{\left|x_j^{m+1}-x_{j-1}^{m+1}\right|}{h_j}=\dfrac{q_j^{m+1}}{h_j}$ we may continue with

$$\begin{aligned}\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^{m}\right)&=-\left(\left|x_u^{m+1}\right|-\frac{q_j^{m+1}}{h_j}\right)\left|y^{m+1}\right|^2+\left(y_u^{m+1},\tau_j^{m+1}-\tau^{m+1}\right)\\&\quad+\left(\frac{q_{j+1}^{m+1}-q_j^{m+1}}{4h_j}+\frac{q_j^{m+1}}{2h_j}\right)\left|y_j^{m+1}\right|^2\\&\quad+\left(\frac{q_{j-1}^{m+1}-q_j^{m+1}}{4h_j}+\frac{q_j^{m+1}}{2h_j}\right)\left|y_{j-1}^{m+1}\right|^2-\frac{q_j^{m+1}}{h_j}\left|y^{m+1}\right|^2\\&\quad+\frac{1}{h_j}\left(RP_j^{m+1}+RM_{j-1}^{m+1}\right)+\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^{m}-\tau_j^{m+1}\right).\end{aligned}$$

A simple calculation leads us finally to

$$\begin{aligned}\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^m\right)=&-\left(\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right)\left|y^{m+1}\right|^2+\left(y_u^{m+1},\tau_j^{m+1}-\tau^{m+1}\right)\\&+\frac{q_{j+1}^{m+1}-q_j^{m+1}}{4h_j}\left|y_j^{m+1}\right|^2+\frac{q_{j-1}^{m+1}-q_j^{m+1}}{4h_j}\left|y_{j-1}^{m+1}\right|^2\\&+\frac{q_j^{m+1}}{2h_j}\left(\left|y_j^{m+1}\right|^2-\left|y^{m+1}\right|^2\right)+\frac{q_j^{m+1}}{2h_j}\left(\left|y_{j-1}^{m+1}\right|^2-\left|y^{m+1}\right|^2\right)\\&+\frac{1}{h_j}\left(RP_j^{m+1}+RM_{j-1}^{m+1}\right)+\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^m-\tau_j^{m+1}\right).\end{aligned}$$

To estimate the last term on the right-hand side of the above equation, we first use (4.48) and then recall (3.25) to obtain in I_j

$$\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^m-\tau_j^{m+1}\right)\leq\left|y_u^{m+1}-y_{hu}^{m+1}\right|\left|\tau_h^{m+1}-\tau_h^m\right|_{|I_j}\leq C\frac{\sqrt{\Delta t}}{h^{\frac{3}{2}}}\left|y_u^{m+1}-y_{hu}^{m+1}\right|_{|I_j}.$$

In view of the above estimate we arrive at

$$\begin{aligned}\left|\left(y_u^{m+1}-y_{hu}^{m+1},\tau_j^m\right)\right|\leq&C\left|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right|_{|I_j}+C\left|\tau^{m+1}-\tau_j^{m+1}\right|\\&+Ch^{-1}\left(\left|q_{j+1}^{m+1}-q_j^{m+1}\right|+\left|q_j^{m+1}-q_{j-1}^{m+1}\right|\right)\\&+C\left(\left|y^{m+1}-y_j^{m+1}\right|+\left|y^{m+1}-y_{j-1}^{m+1}\right|\right)\\&+\frac{C}{h}\left|RP_j^{m+1}+RM_{j-1}^{m+1}\right|+C\frac{\sqrt{\Delta t}}{h^{\frac{3}{2}}}\left|y_u^{m+1}-y_{hu}^{m+1}\right|_{|I_j}.\end{aligned}\tag{4.95}$$

In the following we use the triangle and inverse inequalities as well as the definition of I_h

$$\begin{aligned}\left|q_{j+1}^{m+1}-q_j^{m+1}\right|&=\left|h_{j+1}\left|x_{hu}^{m+1}\right|_{|I_{j+1}}-h_j\left|x_{hu}^{m+1}\right|_{|I_j}\right|\\&\leq h_{j+1}\left|\left|\left(I_hx^{m+1}\right)_u\right|-\left|x_{hu}^{m+1}\right|\right|_{|I_{j+1}}+h_j\left|\left|\left(I_hx^{m+1}\right)_u\right|-\left|x_{hu}^{m+1}\right|\right|_{|I_j}\\&\quad+\left|h_{j+1}\left|\left(I_hx^{m+1}\right)_u\right|_{|I_{j+1}}-h_j\left|\left(I_hx^{m+1}\right)_u\right|_{|I_j}\right|\\&\leq h\left\|\left|\left(I_hx^{m+1}\right)_u\right|-\left|x_{hu}^{m+1}\right|\right\|_{L^\infty(I_j\cup I_{j+1})}\\&\quad+\left|h_{j+1}\frac{x^{m+1}(u_{j+1})-x^{m+1}(u_j)}{h_{j+1}}-h_j\frac{x^{m+1}(u_j)-x^{m+1}(u_{j-1})}{h_j}\right|\\&\leq\sqrt{h}\left\|\left|\left(I_hx^{m+1}\right)_u\right|-\left|x_{hu}^{m+1}\right|\right\|_{L^2(I_j\cup I_{j+1})}+Ch^2.\end{aligned}\tag{4.96}$$

Here, the last estimate follows from the Taylor expansion and smoothness of the continuous solution.
Let us next examine the fifth term on the right-hand side of (4.95). From the smallness of the spatial grid size h and (4.88) we derive

$$\left|RP_j^{m+1}\right|,\left|RM_j^{m+1}\right|\leq C\left(1+h^{-1}\right)\left|R_j^{m+1}\right|+Ch^{-1}\left|R_j^{m+1}\right|^2\leq Ch^{-1}\left|R_j^{m+1}\right|.\tag{4.97}$$

It can be seen that the first summand in (4.97) is of the lowest order. Therefore, one term $\left|R_j^{m+1}\right|$ in the second summand in (4.97) can be estimated by a constant, when one uses the relation (4.79) and an upper bound on the discrete curvature vector from (4.16)

$$\left|R_j^{m+1}\right| = \left|-\alpha_j^{m+1} y_j^{m+1} + \left(\tau_{j+1}^{m+1} - \tau_j^{m+1}\right)\right| \leq Ch + 2. \tag{4.98}$$

Recalling the representation (4.81) of the remainder term and using (2.12) we derive

$$\left|R_j^{m+1}\right| \leq C \sum_{k=0}^{m} \left|\tau_h^{k+1} - \tau_h^k\right|^2_{I_j \cup I_{j+1}} \leq Ch^{-1} \sum_{k=0}^{m} \left\|\tau_h^{k+1} - \tau_h^k\right\|^2_{L^2(I_j \cup I_{j+1})}. \tag{4.99}$$

Putting the calculations (4.96)-(4.99) together and employing (2.12) the inequality (4.95) translates into

$$\begin{aligned}
&\left|\left(y_u^{m+1} - y_{hu}^{m+1}, \tau_j^m\right)\right| \leq Ch + C\sqrt{\Delta t} h^{-2} \left\|y_u^{m+1} - y_{hu}^{m+1}\right\|_{L^2(I_j)} \\
&+ Ch^{-\frac{1}{2}} \left(\left\|y^{m+1} - y_h^{m+1}\right\|_{L^2(I_j)} + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|_{L^2(I_j)} + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|_{L^2(I_j} \right) \\
&+ Ch^{-\frac{1}{2}} \left\|\left|\left(I_h x^{m+1}\right)_u\right| - \left|x_{hu}^{m+1}\right|\right\|_{L^2(I_{j-1} \cup I_j \cup I_{j+1})} + Ch^{-3} \sum_{k=0}^{m} \left\|\tau_h^{k+1} - \tau_h^k\right\|^2_{L^2(I_{j-1} \cup I_j \cup I_{j+1})}.
\end{aligned}$$

Finally, all the expressions in the above inequality have to be squared, integrated over I_j and then summed from $j = 1, ..., N$. Using an interpolation estimate and recalling the notation (4.17) we find

$$\begin{aligned}
\int_0^{2\pi} &\left(y_u^{m+1} - y_{hu}^{m+1}, \tau_h^m\right)^2 \leq C\Delta t h^{-3} \left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 + Ch^{-5} RT^{m+1} \\
&+ C\left(h^2 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2 + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2\right).
\end{aligned} \tag{4.100}$$

Combining the above estimate together with (4.92), (4.100) and Lemma 4.7 results in

$$\begin{aligned}
&\left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 \leq 8C_0 \left(8\delta + C\Delta t + 3\Delta t^2 + C\Delta t h^{-3}\right) \left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 \\
&+ 8C_0 \left(\varepsilon + C_\delta \Delta t^2 h^{-4}\right) \left\|\frac{e_h^{m+1} - e_h^m}{\Delta t}\right\|^2 + C_\delta \left(h^{-1} + h^{-5}\right) RT^{m+1} \\
&+ \left(C_\delta + C_\varepsilon\right) \left(h^2 + \Delta t^2 + \|x^m - x_h^m\|^2 + \|y^m - y_h^m\|^2 + \||x_u^m| - |x_{hu}^m|\|^2\right. \\
&\qquad \left. + \left\|x^{m+1} - x_h^{m+1}\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2 + \left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2\right)
\end{aligned}$$

Without lost of generality we may assume $\Delta t \leq 1$. Imposing the following condition

$$8C_0 \left(8\delta + C\Delta t + C\Delta t h^{-3}\right) \leq \frac{3}{4}$$

we choose δ so small that $64C_0\delta = \dfrac{1}{4}$ and fix the constant $\omega = \min\left\{1, \dfrac{1}{32CC_0}\right\}$. This implies $C\Delta t \leq C\omega h^3 \leq Ch^3 \leq \dfrac{1}{4}$, provided h_0 is sufficiently small. The claim follows.

□

4.7 Discrete length element

The most difficult part in the error analysis is to estimate the error in the length element. From (1.2) follows that $(x_t, \tau) = 0$. Using (4.70) and the smoothness of the continuous solution we obtain further

$$\begin{aligned}
\frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} &= \frac{(\tau^{m+1}, x_u^{m+1}) - (\tau^m, x_u^m)}{\Delta t} = \left(\tau^{m+1}, \frac{x_u^{m+1} - x_u^m}{\Delta t}\right) + \left(\frac{\tau^{m+1} - \tau^m}{\Delta t}, x_u^m\right) \\
&= \left(\tau^{m+1}, \frac{x^{m+1} - x^m}{\Delta t}\right)_u - \left(\tau_u^{m+1}, \frac{x^{m+1} - x^m}{\Delta t}\right) + \left(\frac{\tau^{m+1} - \tau^m}{\Delta t}, x_u^m\right) \\
&= \left(\tau^{m+1}, x_t^{m+1}\right)_u + \left(\tau^{m+1}, \frac{x^{m+1} - x^m}{\Delta t} - x_t^{m+1}\right)_u - \left(y^{m+1}, \frac{x^{m+1} - x^m}{\Delta t}\right)|x_u^{m+1}| \\
&\quad + \left(\frac{\tau^{m+1} - \tau^m}{\Delta t}, x_u^m\right) \\
&= \left(\tau^{m+1}, \frac{x^{m+1} - x^m}{\Delta t} - x_t^{m+1}\right)_u - \left(y^{m+1}, \frac{x^{m+1} - x^m}{\Delta t}\right)|x_u^{m+1}| + \left(\frac{\tau^{m+1} - \tau^m}{\Delta t}, x_u^m\right).
\end{aligned}$$

And finally

$$\begin{aligned}
\frac{|x_u^{m+1}| - |x_u^m|}{\Delta t} + \left(y^{m+1}, \frac{x^{m+1} - x^m}{\Delta t}\right)|x_u^{m+1}| &= \left(\tau^{m+1}, \frac{x^{m+1} - x^m}{\Delta t} - x_t^{m+1}\right)_u \\
&\quad + \left(\frac{\tau^{m+1} - \tau^m}{\Delta t}, x_u^m\right).
\end{aligned} \tag{4.101}$$

In order to derive a discrete analogue of the left-hand side of the equation (4.101), induced on the grid interval I_j, we recall

$$|x_{hu}^m|_{|I_j} = \frac{|x_j^m - x_{j-1}^m|}{h_j} = \frac{q_j^m}{h_j}, \quad m = 0, ..., M$$

and introduce the following lemma.

Lemma 4.11 (Discrete length element)**.** *The discrete length element satisfies:*

$$\begin{aligned}
&\frac{q_j^{m+1} - q_j^m}{\Delta t} + \frac{1}{2}\left(\left(y_j^{m+1}, \frac{x_j^{m+1} - x_j^m}{\Delta t}\right) + \left(y_{j-1}^{m+1}, \frac{x_{j-1}^{m+1} - x_{j-1}^m}{\Delta t}\right)\right) q_j^{m+1} \\
&\qquad = S_j^m - S_{j-1}^m + \left(\frac{\tau_j^{m+1} - \tau_j^m}{\Delta t}, x_j^m - x_{j-1}^m\right) + Rem_j^{m+1}
\end{aligned} \tag{4.102}$$

for $m = 0, ..., M-1$ and $j = 1, ..., N$ with N-periodic indexing and with

$$Rem_j^{m+1} = D_j^{m+1} - E_j^{m+1} + F_j^{m+1} + G_j^{m+1}, \tag{4.103}$$

$$
\begin{aligned}
S_j^m =& -\frac{1}{4\alpha_j^m}\left(\left|y_{j+1}^{m+1}-y_j^{m+1}\right|^2-\left|y_j^{m+1}-y_{j-1}^{m+1}\right|^2\right)+\frac{\alpha_j^{m+1}}{16\alpha_j^m}\left(q_{j+1}^{m+1}-q_j^{m+1}\right)\left|y_j^{m+1}\right|^4\\
&-\frac{\lambda\alpha_j^{m+1}}{4\alpha_j^m}\left(q_{j+1}^{m+1}-q_j^{m+1}\right)\left|y_j^{m+1}\right|^2+\frac{\alpha_j^{m+1}}{16\alpha_j^m}\left|y_j^{m+1}\right|^2\left|y_{j+1}^{m+1}\right|^2\left(q_{j+2}^{m+1}+2q_{j+1}^{m+1}\right)\\
&-\frac{\alpha_j^{m+1}}{16\alpha_j^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2\left(q_{j-1}^{m+1}+2q_j^{m+1}\right),
\end{aligned}
\tag{4.104}
$$

where $D_j^{m+1},...,G_j^{m+1}$ are given by (A.2), (A.17), (A.26) and (A.30), respectively.

Proof. In order to combine the terms on the left-hand side of (4.102), let us represent the first one as a sum of scalar products. To do this, we recall the notations (4.80) and observe

$$
\begin{aligned}
\frac{q_j^{m+1}-q_j^m}{\Delta t} &= \frac{\left|x_j^{m+1}-x_{j-1}^{m+1}\right|-\left|x_j^m-x_{j-1}^m\right|}{\Delta t} = \frac{1}{\Delta t}\frac{\left|x_j^{m+1}-x_{j-1}^{m+1}\right|^2}{\left|x_j^{m+1}-x_{j-1}^{m+1}\right|}-\frac{1}{\Delta t}\frac{\left|x_j^m-x_{j-1}^m\right|^2}{\left|x_j^m-x_{j-1}^m\right|}\\
&= \frac{1}{\Delta t}\left(\frac{x_j^{m+1}-x_{j-1}^{m+1}}{\left|x_j^{m+1}-x_{j-1}^{m+1}\right|},x_j^{m+1}-x_{j-1}^{m+1}\right)-\frac{1}{\Delta t}\left(\frac{x_j^m-x_{j-1}^m}{\left|x_j^m-x_{j-1}^m\right|},x_j^m-x_{j-1}^m\right)\\
&= \frac{1}{\Delta t}\left(\tau_j^{m+1},x_j^{m+1}-x_{j-1}^{m+1}\right)-\frac{1}{\Delta t}\left(\tau_j^m,x_j^m-x_{j-1}^m\right)\\
&= \left(\tau_j^{m+1},\frac{x_j^{m+1}-x_j^m}{\Delta t}\right)-\left(\tau_j^{m+1},\frac{x_{j-1}^{m+1}-x_{j-1}^m}{\Delta t}\right)+\left(\frac{\tau_j^{m+1}-\tau_j^m}{\Delta t},x_j^m-x_{j-1}^m\right).
\end{aligned}
$$

Thus, in view of the above calculations the left-hand side of (4.102) turns into

$$
\begin{aligned}
&\frac{q_j^{m+1}-q_j^m}{\Delta t}+\frac{1}{2}\left(\left(y_j^{m+1},\frac{x_j^{m+1}-x_j^m}{\Delta t}\right)+\left(y_{j-1}^{m+1},\frac{x_{j-1}^{m+1}-x_{j-1}^m}{\Delta t}\right)\right)q_j^{m+1}\\
&=\left(\tau_j^{m+1},\frac{x_j^{m+1}-x_j^m}{\Delta t}\right)-\left(\tau_j^{m+1},\frac{x_{j-1}^{m+1}-x_{j-1}^m}{\Delta t}\right)\\
&\quad+\frac{1}{2}\left(y_j^{m+1},\frac{x_j^{m+1}-x_j^m}{\Delta t}\right)q_j^{m+1}+\frac{1}{2}\left(y_{j-1}^{m+1},\frac{x_{j-1}^{m+1}-x_{j-1}^m}{\Delta t}\right)q_j^{m+1}\\
&\quad+\left(\frac{\tau_j^{m+1}-\tau_j^m}{\Delta t},x_j^m-x_{j-1}^m\right).
\end{aligned}
\tag{4.105}
$$

The next steps consist in calculating the first four scalar products on the right-hand side of (4.105). In order not to interrupt the proof of the lemma, we present here only the results without giving the lengthy proof. For a detailed derivation, we refer the reader to Appendix A.
Thus, in view of (A.1), (A.16), (A.25) and (A.29) the sum of the first four scalar products on the right-hand side of (4.105), which we denote by SP, can be written as

$$
\begin{aligned}
SP :=& \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left(\left|y_{j+1}^{m+1}\right|^2 - \left|y_j^{m+1}\right|^2\right) - \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_{j+1}^{m+1} - y_j^{m+1}\right|^2 \\
&+ \frac{1}{4}\frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_{j+1}^{m+1}\left|y_{j+1}^{m+1}\right|^2 + \alpha_j^{m+1}\left|y_j^{m+1}\right|^2\right) \\
&- \frac{1}{2}\frac{1}{\alpha_j^m}\left(J_{j+1}^{m+1} - J_j^{m+1}\right) + \frac{1}{4}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2 J_{j+1}^{m+1} - \frac{\lambda}{2}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2 + D_j^{m+1} \\
&- \frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left(\left|y_{j-1}^{m+1}\right|^2 - \left|y_{j-2}^{m+1}\right|^2\right) - \frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1} - y_{j-2}^{m+1}\right|^2 \\
&+ \frac{1}{4}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2 + \alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right) \\
&+ \frac{1}{2}\frac{1}{\alpha_{j-1}^m}\left(J_j^{m+1} - J_{j-1}^{m+1}\right) + \frac{1}{4}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 J_{j-1}^{m+1} - \frac{\lambda}{2}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 - E_j^{m+1} \\
&- \frac{1}{4}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(\left|y_{j+1}^{m+1}\right|^2 - \left|y_j^{m+1}\right|^2\right) + \frac{1}{4}\frac{1}{\alpha_j^m}\left(\left|y_j^{m+1}\right|^2 - \left|y_{j-1}^{m+1}\right|^2\right) \\
&+ \frac{1}{4}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left|y_{j+1}^{m+1} - y_j^{m+1}\right|^2 + \frac{1}{4}\frac{1}{\alpha_j^m}\left|y_j^{m+1} - y_{j-1}^{m+1}\right|^2 \\
&- \frac{1}{8}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 + \alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2\right) \\
&- \frac{1}{8}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_{j+1}^{m+1}\left|y_{j+1}^{m+1}\right|^2 + \alpha_j^{m+1}\left|y_j^{m+1}\right|^2\right) \\
&- \frac{1}{8}q_j^{m+1}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(J_{j+1}^{m+1} + J_j^{m+1}\right) + \frac{\lambda}{2}q_j^{m+1}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 + F_j^{m+1} \\
&- \frac{1}{4}\frac{1}{\alpha_{j-1}^m}\left(\left|y_j^{m+1}\right|^2 - \left|y_{j-1}^{m+1}\right|^2\right) + \frac{1}{4}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(\left|y_{j-1}^{m+1}\right|^2 - \left|y_{j-2}^{m+1}\right|^2\right) \\
&+ \frac{1}{4}\frac{1}{\alpha_{j-1}^m}\left|y_j^{m+1} - y_{j-1}^{m+1}\right|^2 + \frac{1}{4}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1} - y_{j-2}^{m+1}\right|^2 \\
&- \frac{1}{8}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2 + \alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right) \\
&- \frac{1}{8}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 + \alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2\right) \\
&- \frac{1}{8}q_j^{m+1}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(J_j^{m+1} + J_{j-1}^{m+1}\right) + \frac{\lambda}{2}q_j^{m+1}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 + G_j^{m+1} \\
=& \sum_{i=1}^{32} P_i.
\end{aligned}
$$

Next, we rearrange the terms in SP according to

$$\begin{aligned}SP =&(P_1+P_{15})+P_4+P_{11}+(P_8+P_{25})+P_{16}+P_{24}+(P_9+P_{27})+(P_2+P_{17})\\&+(P_{18}+P_{26})+(P_3+P_{20})+(P_{10}+P_{28})+(P_{19}+P_{29})+P_5+P_{12}\\&+P_{21}+P_{30}+(P_6+P_{13})+(P_{22}+P_{31})+\left(D_j^{m+1}-E_j^{m+1}+F_j^{m+1}+G_j^{m+1}\right).\end{aligned}$$

The result reads as follows

$$\begin{aligned}SP =&\frac{1}{\alpha_j^m}\left(\frac{\alpha_j^{m+1}}{2q_{j+1}^{m+1}}-\frac{q_j^{m+1}}{4q_{j+1}^{m+1}}\right)\left(\left|y_{j+1}^{m+1}\right|^2-\left|y_j^{m+1}\right|^2\right)-\frac{1}{2}\frac{1}{\alpha_j^m}\left(J_{j+1}^{m+1}-J_j^{m+1}\right)\\
&+\frac{1}{2}\frac{1}{\alpha_{j-1}^m}\left(J_j^{m+1}-J_{j-1}^{m+1}\right)-\frac{1}{\alpha_{j-1}^m}\left(\frac{\alpha_{j-1}^{m+1}}{2q_{j-1}^{m+1}}-\frac{q_j^{m+1}}{4q_{j-1}^{m+1}}\right)\left(\left|y_{j-1}^{m+1}\right|^2-\left|y_{j-2}^{m+1}\right|^2\right)\\
&+\frac{1}{4}\frac{1}{\alpha_j^m}\left(\left|y_j^{m+1}\right|^2-\left|y_{j-1}^{m+1}\right|^2\right)-\frac{1}{4}\frac{1}{\alpha_{j-1}^m}\left(\left|y_j^{m+1}\right|^2-\left|y_{j-1}^{m+1}\right|^2\right)\\
&-\frac{1}{\alpha_{j-1}^m}\left(\frac{\alpha_{j-1}^{m+1}}{2q_{j-1}^{m+1}}-\frac{q_j^{m+1}}{4q_{j-1}^{m+1}}\right)\left|y_{j-1}^{m+1}-y_{j-2}^{m+1}\right|^2\\
&-\frac{1}{\alpha_j^m}\left(\frac{\alpha_j^{m+1}}{2q_{j+1}^{m+1}}-\frac{q_j^{m+1}}{4q_{j+1}^{m+1}}\right)\left|y_{j+1}^{m+1}-y_j^{m+1}\right|^2+\frac{1}{4}\left(\frac{1}{\alpha_j^m}+\frac{1}{\alpha_{j-1}^m}\right)\left|y_j^{m+1}-y_{j-1}^{m+1}\right|^2\\
&+\frac{\alpha_j^{m+1}}{2\alpha_j^m}\left(\frac{\alpha_j^{m+1}}{2q_{j+1}^{m+1}}-\frac{q_j^{m+1}}{4q_{j+1}^{m+1}}\right)\left|y_j^{m+1}\right|^2\left(\alpha_{j+1}^{m+1}\left|y_{j+1}^{m+1}\right|^2+\alpha_j^{m+1}\left|y_j^{m+1}\right|^2\right)\\
&+\frac{\alpha_{j-1}^{m+1}}{2\alpha_{j-1}^m}\left(\frac{\alpha_{j-1}^{m+1}}{2q_{j-1}^{m+1}}-\frac{q_j^{m+1}}{4q_{j-1}^{m+1}}\right)\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2+\alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right)\\
&-\frac{1}{8}\left(\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2+\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\right)\left(\alpha_j^{m+1}\left|y_j^{m+1}\right|^2+\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2\right)\\
&+\frac{1}{4}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2J_{j+1}^{m+1}+\frac{1}{4}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2J_{j-1}^{m+1}\\
&-\frac{1}{8}q_j^{m+1}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(J_{j+1}^{m+1}+J_j^{m+1}\right)-\frac{1}{8}q_j^{m+1}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(J_j^{m+1}+J_{j-1}^{m+1}\right)\\
&-\frac{\lambda}{2}\left(\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2+\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\right)\\
&+\frac{\lambda}{2}q_j^{m+1}\left(\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2+\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\right)+Rem_j^{m+1}.\end{aligned}$$

Recalling the definition (4.80) we find that

$$\alpha_j^{m+1}=\frac{1}{2}\left(q_j^{m+1}+q_{j+1}^{m+1}\right),\quad \alpha_{j-1}^{m+1}=\frac{1}{2}\left(q_{j-1}^{m+1}+q_j^{m+1}\right)$$

and further derive

$$\frac{\alpha_j^{m+1}}{2q_{j+1}^{m+1}} - \frac{q_j^{m+1}}{4q_{j+1}^{m+1}} = \frac{2\alpha_j^{m+1} - q_j^{m+1}}{4q_{j+1}^{m+1}} = \frac{1}{4}, \qquad \frac{\alpha_{j-1}^{m+1}}{2q_{j-1}^{m+1}} - \frac{q_j^{m+1}}{4q_{j-1}^{m+1}} = \frac{2\alpha_{j-1}^{m+1} - q_j^{m+1}}{4q_{j-1}^{m+1}} = \frac{1}{4}.$$

Using the above calculations and in view of the definition (A.4) of J_j^{m+1} we obtain for SP

$$\begin{aligned}
SP =& \frac{1}{4\alpha_j^m}\left(\left|y_{j+1}^{m+1}\right|^2 - \left|y_j^{m+1}\right|^2\right) + \frac{1}{4\alpha_j^m}\left(\left|y_{j-1}^{m+1}\right|^2 - \left|y_{j+1}^{m+1}\right|^2\right)\\
&- \frac{1}{4\alpha_{j-1}^m}\left(\left|y_{j-2}^{m+1}\right|^2 - \left|y_j^{m+1}\right|^2\right) - \frac{1}{4\alpha_{j-1}^m}\left(\left|y_{j-1}^{m+1}\right|^2 - \left|y_{j-2}^{m+1}\right|^2\right)\\
&+ \frac{1}{4}\frac{1}{\alpha_j^m}\left(\left|y_j^{m+1}\right|^2 - \left|y_{j-1}^{m+1}\right|^2\right) - \frac{1}{4}\frac{1}{\alpha_{j-1}^m}\left(\left|y_j^{m+1}\right|^2 - \left|y_{j-1}^{m+1}\right|^2\right)\\
&- \frac{1}{4\alpha_{j-1}^m}\left|y_{j-1}^{m+1} - y_{j-2}^{m+1}\right|^2 - \frac{1}{4\alpha_j^m}\left|y_{j+1}^{m+1} - y_j^{m+1}\right|^2 + \frac{1}{4}\left(\frac{1}{\alpha_j^m} + \frac{1}{\alpha_{j-1}^m}\right)\left|y_j^{m+1} - y_{j-1}^{m+1}\right|^2\\
&+ \frac{\alpha_j^{m+1}}{8\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_{j+1}^{m+1}\left|y_{j+1}^{m+1}\right|^2 + \alpha_j^{m+1}\left|y_j^{m+1}\right|^2\right)\\
&+ \frac{\alpha_{j-1}^{m+1}}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2 + \alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right)\\
&- \frac{1}{8}\left(\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 + \frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\right)\left(\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 + \alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2\right)\\
&+ \frac{1}{8}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\left|y_j^{m+1}\right|^2 + \left|y_{j+1}^{m+1}\right|^2\right)\\
&+ \frac{1}{8}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\left|y_{j-2}^{m+1}\right|^2 + \left|y_{j-1}^{m+1}\right|^2\right)\\
&- \frac{q_j^{m+1}}{16}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\left|y_{j-1}^{m+1}\right|^2 + 2\left|y_j^{m+1}\right|^2 + \left|y_{j+1}^{m+1}\right|^2\right)\\
&- \frac{q_j^{m+1}}{16}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\left|y_{j-2}^{m+1}\right|^2 + 2\left|y_{j-1}^{m+1}\right|^2 + \left|y_j^{m+1}\right|^2\right)\\
&- \frac{\lambda}{2}\left(\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2 + \frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\right)\\
&+ \frac{\lambda}{2}q_j^{m+1}\left(\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 + \frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\right) + Rem_j^{m+1}.
\end{aligned}$$

We observe, that sum of the first six terms in SP results in zero. Combining together the tenth, eleventh and twelfth terms on the right-hand side of the above equation we get

$$
\begin{aligned}
&\frac{\alpha_j^{m+1}}{8\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_{j+1}^{m+1}\left|y_{j+1}^{m+1}\right|^2+\alpha_j^{m+1}\left|y_j^{m+1}\right|^2\right)\\
&+\frac{\alpha_{j-1}^{m+1}}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2+\alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right)\\
&-\frac{1}{8}\left(\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2+\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\right)\left(\alpha_j^{m+1}\left|y_j^{m+1}\right|^2+\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2\right)\\
&=\frac{\alpha_j^{m+1}\alpha_{j+1}^{m+1}}{8\alpha_j^m}\left|y_j^{m+1}\right|^2\left|y_{j+1}^{m+1}\right|^2+\frac{\alpha_{j-1}^{m+1}\alpha_{j-2}^{m+1}}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left|y_{j-2}^{m+1}\right|^2\\
&\quad-\frac{\alpha_{j-1}^{m+1}\alpha_j^{m+1}}{8\alpha_j^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2-\frac{\alpha_{j-1}^{m+1}\alpha_j^{m+1}}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2.
\end{aligned}
$$

A further simplification of SP leads to

$$
\begin{aligned}
SP=&-\frac{1}{4\alpha_{j-1}^m}\left|y_{j-1}^{m+1}-y_{j-2}^{m+1}\right|^2-\frac{1}{4\alpha_j^m}\left|y_{j+1}^{m+1}-y_j^{m+1}\right|^2\\
&+\frac{1}{4\alpha_j^m}\left|y_j^{m+1}-y_{j-1}^{m+1}\right|^2+\frac{1}{4\alpha_{j-1}^m}\left|y_j^{m+1}-y_{j-1}^{m+1}\right|^2\\
&+\frac{\alpha_j^{m+1}\alpha_{j+1}^{m+1}}{8\alpha_j^m}\left|y_j^{m+1}\right|^2\left|y_{j+1}^{m+1}\right|^2+\frac{\alpha_{j-1}^{m+1}\alpha_{j-2}^{m+1}}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left|y_{j-2}^{m+1}\right|^2\\
&-\frac{\alpha_{j-1}^{m+1}\alpha_j^{m+1}}{8\alpha_j^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2-\frac{\alpha_{j-1}^{m+1}\alpha_j^{m+1}}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2\\
&+\frac{\left(\alpha_j^{m+1}\right)^2}{8\alpha_j^m}\left|y_j^{m+1}\right|^4+\frac{\left(\alpha_j^{m+1}\right)^2}{8\alpha_j^m}\left|y_j^{m+1}\right|^2\left|y_{j+1}^{m+1}\right|^2\\
&+\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left|y_{j-2}^{m+1}\right|^2+\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^4\\
&-\frac{\alpha_j^{m+1}q_j^{m+1}}{16\alpha_j^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2-\frac{\alpha_j^{m+1}q_j^{m+1}}{8\alpha_j^m}\left|y_j^{m+1}\right|^4-\frac{\alpha_j^{m+1}q_j^{m+1}}{16\alpha_j^m}\left|y_j^{m+1}\right|^2\left|y_{j+1}^{m+1}\right|^2\\
&-\frac{\alpha_{j-1}^{m+1}q_j^{m+1}}{16\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left|y_{j-2}^{m+1}\right|^2-\frac{\alpha_{j-1}^{m+1}q_j^{m+1}}{8\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^4-\frac{\alpha_{j-1}^{m+1}q_j^{m+1}}{16\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2\\
&-\frac{\lambda\left(\alpha_j^{m+1}\right)^2}{2\alpha_j^m}\left|y_j^{m+1}\right|^2-\frac{\lambda\left(\alpha_{j-1}^{m+1}\right)^2}{2\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\\
&+\frac{\lambda\alpha_j^{m+1}q_j^{m+1}}{2\alpha_j^m}\left|y_j^{m+1}\right|^2+\frac{\lambda\alpha_{j-1}^{m+1}q_j^{m+1}}{2\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2+Rem_j^{m+1}\\
=&\sum_{i=1}^{23}\tilde{P}_i.
\end{aligned}
$$

In order to obtain the representation (4.102), we combine the terms in the following order

$$\begin{aligned}
S_j^m &= \left(\tilde{P}_2 + \tilde{P}_3\right) + \left(\tilde{P}_9 + \tilde{P}_{14}\right) + \left(\tilde{P}_{19} + \tilde{P}_{21}\right) + \left(\tilde{P}_5 + \tilde{P}_{10} + \tilde{P}_{15}\right) + \left(\tilde{P}_7 + \tilde{P}_{13}\right) \\
&= -\frac{1}{4\alpha_j^m}\left(\left|y_{j+1}^{m+1} - y_j^{m+1}\right|^2 - \left|y_j^{m+1} - y_{j-1}^{m+1}\right|^2\right) + \frac{\alpha_j^{m+1}}{16\alpha_j^m}\left|y_j^{m+1}\right|^4\left(2\alpha_j^{m+1} - 2q_j^{m+1}\right) \\
&\quad - \frac{\lambda\alpha_j^{m+1}}{4\alpha_j^m}\left|y_j^{m+1}\right|^2\left(2\alpha_{j+1}^{m+1} - 2q_j^{m+1}\right) \\
&\quad + \frac{\alpha_j^{m+1}}{16\alpha_j^m}\left|y_j^{m+1}\right|^2\left|y_{j+1}^{m+1}\right|^2\left(2\alpha_{j+1}^{m+1} + 2\alpha_j^{m+1} - q_j^{m+1}\right) \\
&\quad - \frac{\alpha_j^{m+1}}{16\alpha_j^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2\left(2\alpha_{j-1}^{m+1} + q_j^{m+1}\right).
\end{aligned}$$

Employing the definition (4.80) of α_j^m we finally receive

$$\begin{aligned}
S_j^m = &-\frac{1}{4\alpha_j^m}\left(\left|y_{j+1}^{m+1} - y_j^{m+1}\right|^2 - \left|y_j^{m+1} - y_{j-1}^{m+1}\right|^2\right) + \frac{\alpha_j^{m+1}}{16\alpha_j^m}\left(q_{j+1}^{m+1} - q_j^{m+1}\right)\left|y_j^{m+1}\right|^4 \\
&- \frac{\lambda\alpha_j^{m+1}}{4\alpha_j^m}\left(q_{j+1}^{m+1} - q_j^{m+1}\right)\left|y_j^{m+1}\right|^2 + \frac{\alpha_j^{m+1}}{16\alpha_j^m}\left|y_j^{m+1}\right|^2\left|y_{j+1}^{m+1}\right|^2\left(q_{j+2}^{m+1} + 2q_{j+1}^{m+1}\right) \\
&- \frac{\alpha_j^{m+1}}{16\alpha_j^m}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2\left(q_{j-1}^{m+1} + 2q_j^{m+1}\right).
\end{aligned}$$

Organizing the terms as it is done below

$$\begin{aligned}
&\left(\tilde{P}_1 + \tilde{P}_4\right) + \left(\tilde{P}_{12} + \tilde{P}_{17}\right) + \left(\tilde{P}_{20} + \tilde{P}_{22}\right) + \left(\tilde{P}_8 + \tilde{P}_{18}\right) + \left(\tilde{P}_6 + \tilde{P}_{16} + \tilde{P}_{11}\right) \\
&= \frac{1}{4\alpha_{j-1}^m}\left(\left|y_j^{m+1} - y_{j-1}^{m+1}\right|^2 - \left|y_{j-1}^{m+1} - y_{j-2}^{m+1}\right|^2\right) - \frac{\alpha_{j-1}^{m+1}}{16\alpha_{j-1}^m}\left(q_j^{m+1} - q_{j-1}^{m+1}\right)\left|y_{j-1}^{m+1}\right|^4 \\
&\quad + \frac{\lambda\alpha_{j-1}^{m+1}}{4\alpha_{j-1}^m}\left(q_j^{m+1} - q_{j-1}^{m+1}\right)\left|y_{j-1}^{m+1}\right|^2 - \frac{\alpha_{j-1}^{m+1}}{16\alpha_{j-1}^m}\left|y_j^{m+1}\right|^2\left|y_{j-1}^{m+1}\right|^2\left(q_{j+1}^{m+1} + 2q_j^{m+1}\right) \\
&\quad + \frac{\alpha_{j-1}^{m+1}}{16\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left|y_{j-2}^{m+1}\right|^2\left(q_{j-2}^{m+1} + 2q_{j-1}^{m+1}\right) = -S_{j-1}^m
\end{aligned}$$

we obtain the claim of the lemma.

□

4.8 Discrete Gronwall's argument

Here, we present the results for the last lemma.

Lemma 4.12 (Discrete Gronwall's argument)**.** *Suppose (4.16) holds. Then there exists* $\Delta t_0 > 0$*, such that for all* $\Delta t \le \Delta t_0$ *we have for* $m = 0, ..., M-1$

$$\left\| \left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right| \right\|^2 \le C\left(h^2 + \Delta t^2\right) + C\Delta t h^{-2} ST^{m+1} + Ch^{-1}RT^{m+1}$$
$$+ C\sum_{k=0}^{m} \Delta t \left(\left\|x^{k+1} - x_h^{k+1}\right\|^2 + \left\|y^{k+1} - y_h^{k+1}\right\|^2 + \left(1 + \frac{\Delta t^2}{h^4}\right) \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|^2 + \frac{RT^{k+1}}{h^5} \right).$$

Proof. For brevity, we denote $\tilde{x}^k = I_h x^k, \tilde{y}^k = I_h y^k$ and introduce

$$\tilde{x}_j^k = x^k(u_j),\ \tilde{y}_j^k = y^k(u_j),\ \tilde{q}_j^k = \left|\tilde{x}_j^k - \tilde{x}_{j-1}^k\right|,$$
$$\tilde{\alpha}_j^k = \frac{1}{2}\left(\tilde{q}_j^k + \tilde{q}_{j+1}^k\right),\ \tilde{\tau}_j^k = \frac{\tilde{x}_j^k - \tilde{x}_{j-1}^k}{\left|\tilde{x}_j^k - \tilde{x}_{j-1}^k\right|}.$$

In view of (4.101) and Lemma 4.11 we obtain on the subinterval I_j for $k = 0, ..., m$

$$\frac{\left|x_u^{k+1}\right| - \left|x_u^k\right|}{\Delta t} - \frac{\left|x_{hu}^{k+1}\right| - \left|x_{hu}^k\right|}{\Delta t} = -\left(\frac{x^{k+1} - x^k}{\Delta t}, y^{k+1}\right)\left|x_u^{k+1}\right| - \frac{1}{h_j}\left(S_j^k - S_{j-1}^k\right)$$
$$+ \frac{q_j^{k+1}}{2h_j}\left(\left(\frac{x_j^{k+1} - x_j^k}{\Delta t}, y_j^{k+1}\right) + \left(\frac{x_{j-1}^{k+1} - x_{j-1}^k}{\Delta t}, y_{j-1}^{k+1}\right)\right) + \left(\frac{\tau^{k+1} - \tau^k}{\Delta t}, x_u^k\right) \tag{4.106}$$
$$+ \left(\tau^{k+1}, \frac{x^{k+1} - x^k}{\Delta t} - x_t^{k+1}\right)_u - \left(\frac{\tau_j^{k+1} - \tau_j^k}{\Delta t}, \frac{x_j^k - x_{j-1}^k}{h_j}\right) - \frac{1}{h_j} Rem_j^{k+1}.$$

Next, we multiply the equation (4.106) by Δt and then sum from $k = 0, .., m$. Using the reverse triangle inequality and notations introduced above we arrive at

$$\left(\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right) - \left|x_u^0 - x_{hu}^0\right|$$
$$\le \frac{1}{2h_j}\sum_{k=0}^{m} \Delta t \left| \left(\left(\frac{\tilde{x}_j^{k+1} - \tilde{x}_j^k}{\Delta t}, \tilde{y}_j^{k+1}\right) + \left(\frac{\tilde{x}_{j-1}^{k+1} - \tilde{x}_{j-1}^k}{\Delta t}, \tilde{y}_{j-1}^{k+1}\right)\right) \tilde{q}_j^{k+1} \right.$$
$$\left. - \left(\left(\frac{x_j^{k+1} - x_j^k}{\Delta t}, y_j^{k+1}\right) + \left(\frac{x_{j-1}^{k+1} - x_{j-1}^k}{\Delta t}, y_{j-1}^{k+1}\right)\right) q_j^{k+1} \right|$$
$$+ \sum_{k=0}^{m} \Delta t \left| \frac{1}{2}\left(\left(\frac{\tilde{x}_j^{k+1} - \tilde{x}_j^k}{\Delta t}, \tilde{y}_j^{k+1}\right) + \left(\frac{\tilde{x}_{j-1}^{k+1} - \tilde{x}_{j-1}^k}{\Delta t}, \tilde{y}_{j-1}^{k+1}\right)\right) \frac{\tilde{q}_j^{k+1}}{h_j} \right.$$
$$\left. - \left(\frac{x^{k+1} - x^k}{\Delta t}, y^{k+1}\right)\left|x_u^{k+1}\right| \right|$$
$$+ \sum_{k=0}^{m} \Delta t \left(\tau^{k+1}, \left(\frac{x^{k+1} - x^k}{\Delta t}\right) - x_t^{k+1}\right)_u + \sum_{k=0}^{m} \Delta t \left(\frac{\tau^{k+1} - \tau^k}{\Delta t}, x_u^k\right)$$
$$- \frac{1}{h_j}\sum_{k=0}^{m} \Delta t \left(S_j^k - S_{j-1}^k\right) - \sum_{k=0}^{m} \Delta t \left(\frac{\tau_j^{k+1} - \tau_j^k}{\Delta t}, \frac{x_j^k - x_{j-1}^k}{h_j}\right) - \frac{1}{h_j}\sum_{k=0}^{m} \Delta t Rem_j^{k+1}$$
$$= I + ... + VII.$$

We recall that $q_j^k = \left|x_j^k - x_{j-1}^k\right| = h_j \left|x_{hu}^k\right|_{I_j}$ and from (4.8), (4.16) we deduce on I_j

$$\begin{aligned}&\frac{1}{2}c_0 h_j \leq q_j^k \leq 2C_0 h_j, \quad k = 0, ..., m,\\&\frac{1}{4}c_0 h_j \leq q_j^{m+1} \leq 4C_0 h_j.\end{aligned}$$

Let us next estimate the expressions $I, ..., VII$. To begin, we split the first term in the following way

$$\begin{aligned}I \leq & \frac{1}{2h_j}\sum_{k=0}^{m}\Delta t \left| \left(\left(\frac{\tilde{x}_j^{k+1} - \tilde{x}_j^k}{\Delta t}, \tilde{y}_j^{k+1} \right) + \left(\frac{\tilde{x}_{j-1}^{k+1} - \tilde{x}_{j-1}^k}{\Delta t}, \tilde{y}_{j-1}^{k+1} \right) \right) \left(\tilde{q}_j^{k+1} - q_j^{k+1}\right) \right|\\&+ \sum_{k=0}^{m}\Delta t \frac{q_j^{k+1}}{2h_j}\left| \left(\frac{\tilde{x}_j^{k+1} - \tilde{x}_j^k}{\Delta t}, \tilde{y}_j^{k+1} - y_j^{k+1} \right) + \left(\frac{\tilde{x}_j^{k+1} - \tilde{x}_j^k}{\Delta t} - \frac{x_j^{k+1} - x_j^k}{\Delta t}, y_j^{k+1} \right) \right|\\&+ \sum_{k=0}^{m}\Delta t \frac{q_j^{k+1}}{2h_j}\left| \left(\frac{\tilde{x}_{j-1}^{k+1} - \tilde{x}_{j-1}^k}{\Delta t}, \tilde{y}_{j-1}^{k+1} - y_{j-1}^{k+1} \right) \right.\\&\qquad \left. + \left(\frac{\tilde{x}_{j-1}^{k+1} - \tilde{x}_{j-1}^k}{\Delta t} - \frac{x_{j-1}^{k+1} - x_{j-1}^k}{\Delta t}, y_{j-1}^{k+1} \right) \right|.\end{aligned}$$

Further, in view of the smoothness of the continuous solution and bounds (4.16) we estimate

$$\begin{aligned}I \leq & \frac{C}{h_j}\sum_{k=0}^{m}\Delta t \left|\tilde{q}_j^{k+1} - q_j^{k+1}\right| + C\sum_{k=0}^{m}\Delta t \left(\left|\tilde{y}_j^{k+1} - y_j^{k+1}\right| + \left|\tilde{y}_{j-1}^{k+1} - y_{j-1}^{k+1}\right|\right)\\&+ C\sum_{k=0}^{m}\Delta t \left(\left| \frac{\tilde{x}_j^{k+1} - \tilde{x}_j^k}{\Delta t} - \frac{x_j^{k+1} - x_j^k}{\Delta t} \right| + \left| \frac{\tilde{x}_{j-1}^{k+1} - \tilde{x}_{j-1}^k}{\Delta t} - \frac{x_{j-1}^{k+1} - x_{j-1}^k}{\Delta t} \right| \right).\end{aligned}$$

Using now (2.12), an inverse assumption on h and recalling the definition of e_h^k we get

$$\begin{aligned}I \leq & C\sum_{k=0}^{m}\Delta t \left(\left\| \left|\tilde{x}_{hu}^{k+1}\right| - \left|x_{hu}^{k+1}\right| \right\|_{L^\infty(I_j)} + \left\|\tilde{y}_h^{k+1} - y_h^{k+1}\right\|_{L^\infty(I_j)} \right)\\&+ C\sum_{k=0}^{m}\Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|_{L^\infty(I_j)}\\\leq & \frac{C}{\sqrt{h}}\sum_{k=0}^{m}\Delta t \left(\left\| \left|x_u^{k+1}\right| - \left|x_{hu}^{k+1}\right| \right\|_{L^2(I_j)} + \left\|y^{k+1} - y_h^{k+1}\right\|_{L^2(I_j)} \right)\\&+ \frac{C}{\sqrt{h}}\sum_{k=0}^{m}\Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|_{L^2(I_j)} + Ch.\end{aligned} \tag{4.107}$$

Analogous calculations lead to

$$
\begin{aligned}
II \leq & \sum_{k=0}^{m} \Delta t \left| \frac{1}{2} \left(\left(\frac{\tilde{x}_j^{k+1} - \tilde{x}_j^k}{\Delta t}, \tilde{y}_j^{k+1} \right) + \left(\frac{\tilde{x}_{j-1}^{k+1} - \tilde{x}_{j-1}^k}{\Delta t}, \tilde{y}_{j-1}^{k+1} \right) \right) \frac{\left| \tilde{x}_j^{k+1} - \tilde{x}_{j-1}^{k+1} \right|}{h_j} \right. \\
& \qquad\qquad \left. - \left(\frac{x^{k+1} - x^k}{\Delta t}, y^{k+1} \right) \left| x_u^{k+1} \right| \right| \\
\leq & \sum_{k=0}^{m} \Delta t \left| \left(\frac{x^{k+1} - x^k}{\Delta t}, y^{k+1} \right) \left(\frac{\left| \tilde{x}_j^{k+1} - \tilde{x}_{j-1}^{k+1} \right|}{h_j} - \left| x_u^{k+1} \right| \right) \right| \\
& + \sum_{k=0}^{m} \Delta t \frac{\tilde{q}_j^{k+1}}{2h_j} \left| \left(\frac{x^{k+1} - x^k}{\Delta t}, \tilde{y}_j^{k+1} - y^{k+1} \right) + \left(\frac{x^{k+1} - x^k}{\Delta t}, \tilde{y}_{j-1}^{k+1} - y^{k+1} \right) \right| \\
& + \sum_{k=0}^{m} \Delta t \frac{\tilde{q}_j^{k+1}}{2h_j} \left| \left(\frac{\tilde{x}_j^{k+1} - \tilde{x}_j^k}{\Delta t} - \frac{x^{k+1} - x^k}{\Delta t}, \tilde{y}_j^{k+1} \right) \right. \\
& \qquad\qquad \left. + \left(\frac{\tilde{x}_{j-1}^{k+1} - \tilde{x}_{j-1}^k}{\Delta t} - \frac{x^{k+1} - x^k}{\Delta t}, \tilde{y}_{j-1}^{k+1} \right) \right| \\
\leq & C \sum_{k=0}^{m} \Delta t \left(\left\| \left(I_h x^{k+1} \right)_u - x_u^{k+1} \right\|_{L^\infty(I_j)} + \left\| I_h y^{k+1} - y^{k+1} \right\|_{L^\infty(I_j)} \right) \\
& + C \sum_{k=0}^{m} \Delta t \left\| I_h \left[\frac{x^{k+1} - x^k}{\Delta t} \right] - \frac{x^{k+1} - x^k}{\Delta t} \right\|_{L^\infty(I_j)} \leq Ch.
\end{aligned}
\tag{4.108}
$$

Further, smoothness of the continuous solution together with $m\Delta t = T$ implies

$$
\begin{aligned}
III = & \sum_{k=0}^{m} \Delta t \left(\tau^{k+1}, \left(\frac{x^{k+1} - x^k}{\Delta t} \right) - x_t^{k+1} \right)_u \leq C \sum_{k=0}^{M} \Delta t^2 \leq C \Delta t, \\
IV = & \sum_{k=0}^{m} \Delta t \left(\frac{\tau^{k+1} - \tau^k}{\Delta t}, x_u^k \right) = \sum_{k=0}^{m} \Delta t \left(\frac{\tau^{k+1} - \tau^k}{\Delta t}, \tau^k \right) \left| x_u^k \right| \\
= & \frac{1}{2} \sum_{k=0}^{m} \Delta t \frac{\left| \tau^{k+1} - \tau^k \right|^2}{\Delta t} \left| x_u^k \right| \leq C \Delta t.
\end{aligned}
\tag{4.109}
$$

From the inverse inequality (2.12) and relation (3.21) we infer

$$
\begin{aligned}
VI = & - \sum_{k=0}^{m} \Delta t \left(\frac{\tau_j^{k+1} - \tau_j^k}{\Delta t}, \frac{x_j^k - x_{j-1}^k}{h_j} \right) \leq \sum_{k=0}^{m} \Delta t \left| \left(\frac{\tau_h^{k+1} - \tau_h^k}{\Delta t}, \tau_h^k \right) \right|_{I_j} \left| x_{hu}^k \right|_{I_j} \\
\leq & \frac{1}{2} \sum_{k=0}^{m} \Delta t \frac{\left| \tau_h^{k+1} - \tau_h^k \right|_{I_j}^2}{\Delta t} \left| x_{hu}^k \right|_{I_j} \leq C h^{-1} \sum_{k=0}^{m} \left\| \tau_h^{k+1} - \tau_h^k \right\|_{L^2(I_j)}^2 .
\end{aligned}
\tag{4.110}
$$

Now we turn our attention to the most difficult part of this lemma − estimation of the terms V and VII. First, we deal with the difference $S_j^k - S_{j-1}^k$. In the following,

we will apply the same technique already used above. In particular, we insert into this difference a continuous equivalent and then consider the difference between corresponding continuous and discrete terms. The idea is to generate the terms which already appear on the right-hand side of the previous estimates.
For the sake of readability, we recall (4.104) and present

$$\begin{aligned} S_j^k = &-\frac{1}{4\alpha_j^k}\left(\left|y_{j+1}^{k+1}-y_j^{k+1}\right|^2-\left|y_j^{k+1}-y_{j-1}^{k+1}\right|^2\right)+\frac{\alpha_j^{k+1}}{16\alpha_j^k}\left(q_{j+1}^{k+1}-q_j^{k+1}\right)\left|y_j^{k+1}\right|^4 \\ &-\frac{\lambda\alpha_j^{k+1}}{4\alpha_j^k}\left(q_{j+1}^{k+1}-q_j^{k+1}\right)\left|y_j^{k+1}\right|^2+\frac{\alpha_j^{k+1}}{16\alpha_j^k}\left|y_j^{k+1}\right|^2\left|y_{j+1}^{k+1}\right|^2\left(q_{j+2}^{k+1}+2q_{j+1}^{k+1}\right) \\ &-\frac{\alpha_j^{k+1}}{16\alpha_j^k}\left|y_{j-1}^{k+1}\right|^2\left|y_j^{k+1}\right|^2\left(q_{j-1}^{k+1}+2q_j^{k+1}\right). \end{aligned}$$

We claim that

$$\left|\tilde{S}_j^k\right| = \left|S_j^k\left(\tilde{q}_j^k,\tilde{q}_{j+1}^k,\tilde{q}_{j-1}^{k+1},\tilde{q}_j^{k+1},\tilde{q}_{j+1}^{k+1},\tilde{q}_{j+2}^{k+1},\tilde{y}_{j-1}^{k+1},\tilde{y}_j^{k+1},\tilde{y}_{j+1}^{k+1}\right)\right| \le Ch^2.$$

Let us demonstrate that each of five terms in $\tilde{S}_j^k$ can be estimated by a multiple of the spatial grid size squared. By means of the reverse triangle inequality, Taylor expansion and smoothness of the continuous we deduce for the first two terms in $\tilde{S}_j^k$

$$\begin{aligned} &\frac{1}{4\tilde{\alpha}_j^k}\left|\left|\tilde{y}_{j+1}^{k+1}-\tilde{y}_j^{k+1}\right|^2-\left|\tilde{y}_j^{k+1}-\tilde{y}_{j-1}^{k+1}\right|^2\right| \le \frac{1}{4\tilde{\alpha}_j^k}\left|\tilde{y}_{j+1}^{k+1}-2\tilde{y}_j^{k+1}+\tilde{y}_{j-1}^{k+1}\right|\left|\tilde{y}_{j+1}^{k+1}-\tilde{y}_{j-1}^{k+1}\right| \\ &\le \frac{C}{h}h^2\left(\left|\tilde{y}_{j+1}^{k+1}-\tilde{y}_j^{k+1}\right|+\left|\tilde{y}_j^{k+1}-\tilde{y}_{j-1}^{k+1}\right|\right) \le Ch^2, \\ &\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{q}_{j+1}^{k+1}-\tilde{q}_j^{k+1}\right|\left|\tilde{y}_j^{k+1}\right|^4 \le C\left|\left|\tilde{x}_{j+1}^{k+1}-\tilde{x}_j^{k+1}\right|-\left|\tilde{x}_j^{k+1}-\tilde{x}_{j-1}^{k+1}\right|\right| \le Ch^2. \end{aligned}$$

From the last estimate we also have

$$\frac{\lambda\tilde{\alpha}_j^{k+1}}{4\tilde{\alpha}_j^k}\left|\tilde{q}_{j+1}^{k+1}-\tilde{q}_j^{k+1}\right|\left|\tilde{y}_j^{k+1}\right|^2 \le Ch^2.$$

The remaining in $\tilde{S}_j^k$ two expressions after adding a zero term can be written as

$$\begin{aligned} &\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{y}_j^{k+1}\right|^2\left|\tilde{y}_{j+1}^{k+1}\right|^2\left(\tilde{q}_{j+2}^{k+1}+2\tilde{q}_{j+1}^{k+1}\right)-\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{y}_{j-1}^{k+1}\right|^2\left|\tilde{y}_j^{k+1}\right|^2\left(\tilde{q}_{j-1}^{k+1}+2\tilde{q}_j^{k+1}\right) \\ &= \frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{y}_j^{k+1}\right|^2\left(\left|\tilde{y}_{j+1}^{k+1}\right|^2-\left|\tilde{y}_j^{k+1}\right|^2\right)\left(\tilde{q}_{j+2}^{k+1}+2\tilde{q}_{j+1}^{k+1}\right) \\ &\quad+\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{y}_j^{k+1}\right|^2\left(\left|\tilde{y}_j^{k+1}\right|^2-\left|\tilde{y}_{j-1}^{k+1}\right|^2\right)\left(\tilde{q}_{j-1}^{k+1}+2\tilde{q}_j^{k+1}\right) \\ &\quad+\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{y}_j^{k+1}\right|^4\left(\tilde{q}_{j+2}^{k+1}+2\tilde{q}_{j+1}^{k+1}-\tilde{q}_{j-1}^{k+1}-2\tilde{q}_j^{k+1}\right). \end{aligned}$$

Observing that

$$\tilde{q}_{j+2}^{k+1}+2\tilde{q}_{j+1}^{k+1}-\tilde{q}_{j-1}^{k+1}-2\tilde{q}_{j}^{k+1}=\left(\tilde{q}_{j+2}^{k+1}-\tilde{q}_{j+1}^{k+1}\right)+3\left(\tilde{q}_{j+1}^{k+1}-\tilde{q}_{j}^{k+1}\right)+\left(\tilde{q}_{j}^{k+1}-\tilde{q}_{j-1}^{k+1}\right)$$

a similar argument may be used to obtain

$$\left|\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{y}_j^{k+1}\right|^2\left|\tilde{y}_{j+1}^{k+1}\right|^2\left(\tilde{q}_{j+2}^{k+1}+2\tilde{q}_{j+1}^{k+1}\right)-\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{y}_{j-1}^{k+1}\right|^2\left|\tilde{y}_j^{k+1}\right|^2\left(\tilde{q}_{j-1}^{k+1}+2\tilde{q}_j^{k+1}\right)\right|\leq Ch^2.$$

Arguing in the same way we may conclude $\tilde{S}_{j-1}^k\leq Ch^2$. Therefore,

$$\left|S_j^k-S_{j-1}^k\right|\leq\left|S_j^k-\tilde{S}_j^k\right|+\left|S_{j-1}^k-\tilde{S}_{j-1}^k\right|+Ch^2. \tag{4.111}$$

Hence, it remains to estimate the difference between S_j^k and $\tilde{S}_j^k$ since the difference $S_{j-1}^k-\ \tilde{S}_{j-1}^k$ behaves in a similar way. We will treat the terms separately.
At first, we derive an inequality which will be often used in the remaining estimates. Thus, from the smoothness of the continuous solution, the inverse inequality (2.12) an interpolation estimate we infer

$$\begin{aligned}\left|q_j^{k+1}-\tilde{q}_j^{k+1}\right|&=\left|\left|x_j^{k+1}-x_{j-1}^{k+1}\right|-\left|\tilde{x}_j^{k+1}-\tilde{x}_{j-1}^{k+1}\right|\right|=h_j\left|\left|x_{hu}^{k+1}\right|-\left|\tilde{x}_{hu}^{k+1}\right|\right|_{I_j}\\&\leq\sqrt{h_j}\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|_{L^2(I_j)}+Ch_j^2.\end{aligned} \tag{4.112}$$

Next, recalling the definition of α_j^k and using (4.112) results in

$$\begin{aligned}&\left|\frac{1}{4\alpha_j^k}\left(\left|y_{j+1}^{k+1}-y_j^{k+1}\right|^2-\left|y_j^{k+1}-y_{j-1}^{k+1}\right|^2\right)-\frac{1}{4\tilde{\alpha}_j^k}\left(\left|\tilde{y}_{j+1}^{k+1}-\tilde{y}_j^{k+1}\right|^2-\left|\tilde{y}_j^{k+1}-\tilde{y}_{j-1}^{k+1}\right|^2\right)\right|\\
&\leq\left|\frac{\tilde{q}_j^{k+1}+\tilde{q}_{j+1}^{k+1}-q_j^{k+1}-q_{j+1}^{k+1}}{8\alpha_j^k\tilde{\alpha}_j^k}\right|\left|\left|\tilde{y}_{j+1}^{k+1}-\tilde{y}_j^{k+1}\right|^2-\left|\tilde{y}_j^{k+1}-\tilde{y}_{j-1}^{k+1}\right|^2\right|\\
&\quad+\frac{1}{4\alpha_j^k}\left|\left(\left|y_{j+1}^{k+1}-y_j^{k+1}\right|^2-\left|y_j^{k+1}-y_{j-1}^{k+1}\right|^2\right)-\left(\left|\tilde{y}_{j+1}^{k+1}-\tilde{y}_j^{k+1}\right|^2-\left|\tilde{y}_j^{k+1}-\tilde{y}_{j-1}^{k+1}\right|^2\right)\right|\\
&\leq C\frac{\left|\tilde{q}_{j+1}^{k+1}-q_{j+1}^{k+1}\right|+\left|\tilde{q}_j^{k+1}-q_j^{k+1}\right|}{h^2}h^2\\
&\quad+\frac{C}{h}\left(h_{j+1}^2\left|\left|\tilde{y}_{hu}^{k+1}\right|^2-\left|y_{hu}^{k+1}\right|^2\right|_{I_{j+1}}+h_j^2\left|\left|\tilde{y}_{hu}^{k+1}\right|^2-\left|y_{hu}^{k+1}\right|^2\right|_{I_j}\right)\\
&\leq\sqrt{h}\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|_{L^2(I_j\cup I_{j+1})}+Ch^2+Ch\left|\tilde{y}_{hu}^{k+1}-y_{hu}^{k+1}\right|_{I_j}\left(\left|\tilde{y}_{hu}^{k+1}\right|_{I_j}+\left|y_{hu}^{k+1}\right|_{I_j}\right)\\
&\quad+Ch\left|\tilde{y}_{hu}^{k+1}-y_{hu}^{k+1}\right|_{I_{j+1}}\left(\left|\tilde{y}_{hu}^{k+1}\right|_{I_{j+1}}+\left|y_{hu}^{k+1}\right|_{I_{j+1}}\right)\\
&\leq\sqrt{h}\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|_{L^2(I_j\cup I_{j+1})}+Ch^2\left(1+\left\|y_{hu}^{k+1}\right\|_{L^\infty}\right)\\
&\quad+C\sqrt{h}\left\|y_u^{k+1}-y_{hu}^{k+1}\right\|_{L^2(I_j\cup I_{j+1})}\left(1+\left\|y_{hu}^{k+1}\right\|_{L^\infty}\right),\end{aligned}$$

where for the last step we have used an interpolation estimate.

Using (4.16) we achieve for the next typical term

$$
\begin{aligned}
&\left|\frac{\alpha_j^{k+1}}{16\alpha_j^k}\left|y_j^{k+1}\right|^2\left|y_{j+1}^{k+1}\right|^2\left(q_{j+2}^{k+1}+2q_{j+1}^{k+1}\right)-\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\left|\tilde{y}_j^{k+1}\right|^2\left|\tilde{y}_{j+1}^{k+1}\right|^2\left(\tilde{q}_{j+2}^{k+1}+2\tilde{q}_{j+1}^{k+1}\right)\right| \\
&\leq\left|\frac{\alpha_j^{k+1}}{16\alpha_j^k}-\frac{\tilde{\alpha}_j^{k+1}}{16\tilde{\alpha}_j^k}\right|\left|\tilde{y}_j^{k+1}\right|^2\left|\tilde{y}_{j+1}^{k+1}\right|^2\left(\tilde{q}_{j+2}^{k+1}+2\tilde{q}_{j+1}^{k+1}\right) \\
&\quad+C\left|\left|y_j^{k+1}\right|^2\left|y_{j+1}^{k+1}\right|^2-\left|\tilde{y}_j^{k+1}\right|^2\left|\tilde{y}_{j+1}^{k+1}\right|^2\right|\left(q_{j+2}^{k+1}+2q_{j+1}^{k+1}\right) \\
&\quad+C\left|\tilde{y}_j^{k+1}\right|^2\left|\tilde{y}_{j+1}^{k+1}\right|^2\left|q_{j+2}^{k+1}-\tilde{q}_{j+2}^{k+1}+2\left(q_{j+1}^{k+1}-\tilde{q}_{j+1}^{k+1}\right)\right| \\
&\leq C\left(\left|\tilde{q}_{j+2}^{k+1}-q_{j+2}^{k+1}\right|+\left|\tilde{q}_{j+1}^{k+1}-q_{j+1}^{k+1}\right|+\left|\tilde{q}_j^{k+1}-q_j^{k+1}\right|+\left|\tilde{q}_{j+1}^{k}-q_{j+1}^{k}\right|+\left|\tilde{q}_j^{k}-q_j^{k}\right|\right) \\
&\quad+Ch\left(\left|\tilde{y}_{j+1}^{k+1}\right|^2\left|\left|\tilde{y}_j^{k+1}\right|^2-\left|y_j^{k+1}\right|^2\right|+\left|y_j^{k+1}\right|^2\left|\left|\tilde{y}_{j+1}^{k+1}\right|^2-\left|y_{j+1}^{k+1}\right|^2\right|\right) \\
&\leq\sqrt{h}\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|_{L^2(I_j\cup I_{j+1}\cup I_{j+2})}+Ch^2+Ch\left\|\tilde{y}_h^{k+1}-y_h^{k+1}\right\|_{L^\infty(I_{j+1})} \\
&\leq\sqrt{h}\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|_{L^2(I_j\cup I_{j+1}\cup I_{j+2})}+Ch^2+C\sqrt{h}\left\|y^{k+1}-y_h^{k+1}\right\|_{L^2(I_{j+1})}.
\end{aligned}
$$

The remaining terms in the difference $S_j^k-\tilde{S}_j^k$ and $S_{j-1}^k-\tilde{S}_{j-1}^k$ can be estimated in a similar way. Thus, introducing a notation

$$
I_{(j)}=I_{j-2}\cup I_{j-1}\cup I_j\cup I_{j+1}\cup I_{j+2} \tag{4.113}
$$

and recalling (4.111) we arrive at

$$
\begin{aligned}
\left|S_j^k-S_{j-1}^k\right|\leq&C\sqrt{h}\left(\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|_{L^2(I_{(j)})}+\left\|y^{k+1}-y_h^{k+1}\right\|_{L^2(I_{(j)})}\right) \\
&+C\left(h^2+\sqrt{h}\left\|y_u^{k+1}-y_{hu}^{k+1}\right\|_{L^2(I_{(j)})}\right)\left(1+\left\|y_{hu}^{k+1}\right\|_{L^\infty}\right).
\end{aligned} \tag{4.114}
$$

Combining the estimates (4.107)-(4.110) for the terms I, II, III, IV and VI with the estimate (4.114) for V we get in I_j

$$
\begin{aligned}
&\left|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right|_{|I_j}\leq\left|x_u^0-x_{hu}^0\right|+C\left(h+\Delta t\right)+Ch^{-1}\sum_{k=0}^m\left\|\tau_h^{k+1}-\tau_h^k\right\|_{L^2(I_j)}^2 \\
&+\frac{C}{\sqrt{h}}\sum_{k=0}^m\Delta t\left(\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|_{L^2(I_{(j)})}+\left\|y^{k+1}-y_h^{k+1}\right\|_{L^2(I_{(j)})}+\left\|\frac{e_h^{k+1}-e_h^k}{\Delta t}\right\|_{L^2(I_{(j)})}\right) \\
&+C\sum_{k=0}^m\Delta t\left(h+\frac{1}{\sqrt{h}}\left\|y_u^{k+1}-y_{hu}^{k+1}\right\|_{L^2(I_{(j)})}\right)\left(1+\left\|y_{hu}^{k+1}\right\|_{L^\infty}\right)+\frac{1}{h_j}\sum_{k=0}^m\Delta t\left|Rem_j^{k+1}\right|.
\end{aligned}
$$

In order to obtain the claim of the lemma, we have to square the above result, integrate it over I_j and then sum from $j=1,...,N$. For simplicity, we perform this step separately for the last term on the right-hand side of the above inequality. Using the auxiliary result (B.1), an inverse assumption on h and Cauchy's inequality we find

$$\sum_{j=1}^{N}\int_{I_j}\frac{1}{h_j^2}\left(\sum_{k=0}^{m}\Delta t h^{-2}\left(1+\left\|y_{hu}^{k+1}\right\|_{L^\infty}\right)\sum_{l=0}^{k}\left\|\tau_h^{l+1}-\tau_h^{l}\right\|_{L^2(I_{(j)})}^2\right)^2$$
$$+\sum_{j=1}^{N}\int_{I_j}\frac{1}{h_j^2}\left(\sum_{k=0}^{m}\Delta t h^{-\frac{1}{2}}\left\|y_{hu}^{k+1}\right\|_{L^\infty}\left\|\tau_h^{k+1}-\tau_h^{k}\right\|_{L^2(I_{(j)})}\right)^2$$
$$\leq C\sum_{k=0}^{m}\Delta t\left(1+\left\|y_{hu}^{k+1}\right\|_{L^\infty}^2\right)\sum_{k=0}^{m}\Delta t h^{-5}\left(\sum_{l=0}^{k}\left\|\tau_h^{l+1}-\tau_h^{l}\right\|^2\right)^2$$
$$+C\sum_{k=0}^{m}\Delta t\left(1+\left\|y_{hu}^{k+1}\right\|_{L^\infty}^2\right)\sum_{k=0}^{m}\Delta t h^{-2}\left\|\tau_h^{k+1}-\tau_h^{k}\right\|^2.$$

Denoting for brevity

$$ST^{m+1}:=\sum_{k=0}^{m}\left\|\tau_h^{k+1}-\tau_h^{k}\right\|^2 \tag{4.115}$$

and recalling the notation (4.17) together with the initial condition $x_h^0=I_hx^0$, we obtain with the help of Cauchy's inequality

$$\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|^2\leq C\left(h^2+\Delta t^2\right)+C\Delta t h^{-2}ST^{m+1}+Ch^{-1}RT^{m+1}$$
$$+C\sum_{k=0}^{m}\Delta t\left(\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|^2+\left\|y^{k+1}-y_h^{k+1}\right\|^2+\left\|\frac{e_h^{k+1}-e_h^{k}}{\Delta t}\right\|^2\right)$$
$$+C\sum_{k=0}^{m}\Delta t\left(1+\left\|y_{hu}^{k+1}\right\|_{L^\infty}^2\right)\left(h^2\sum_{k=0}^{m}\Delta t+\sum_{k=0}^{m}\Delta t\left\|y_u^{k+1}-y_{hu}^{k+1}\right\|^2\right)+Ch^{-5}\sum_{k=0}^{m}\Delta t RT^{k+1}.$$

Further, Lemma 4.10 with $\varepsilon=1$ and a priori bounds (4.16) imply

$$\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|^2\leq C\left(h^2+\Delta t^2\right)+C\Delta t h^{-2}ST^{m+1}+Ch^{-1}RT^{m+1}$$
$$+Ch^{-5}\sum_{k=0}^{m}\Delta t RT^{k+1}+C\sum_{k=0}^{m}\Delta t\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|^2$$
$$+C\sum_{k=0}^{m}\Delta t\left(\left\|x^{k+1}-x_h^{k+1}\right\|^2+\left\|y^{k+1}-y_h^{k+1}\right\|^2+\left(1+\Delta t^2h^{-4}\right)\left\|\frac{e_h^{k+1}-e_h^{k}}{\Delta t}\right\|^2\right).$$

Next, we split the fifth term on the right-hand side of the above inequality as follows

$$\Delta t\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|^2+\sum_{k=0}^{m-1}\Delta t\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|^2.$$

Finally, smallness of Δt allows us to move the term $\Delta t\left\|\left|x_u^{m+1}\right|-\left|x_{hu}^{m+1}\right|\right\|^2$ to the left-hand side, provided Δt_0 is small enough. To complete the proof, the discrete Gronwall's Lemma (2.2) applies and yields the result.

□

Chapter 5

Completion of Induction argument. A posteriori estimates

5.1 Completion of Induction argument

First of all, we start with an auxiliary result, which will be used later.

Lemma 5.1. *Suppose (4.16) holds. Then there exists $\omega > 0$ independent of h and Δt, such that for all $\Delta t \le \omega h^2$ we have for $m = 0, ..., M-1$*

$$\frac{c_0}{64}\sum_{k=0}^{m}\Delta t\left\|\frac{e_h^{k+1}-e_h^k}{\Delta t}\right\|^2 + \frac{c_0}{32}\left\|y^{m+1}-y_h^{m+1}\right\|^2 \le C\left(h^2+\Delta t^2\right) + C\frac{\Delta t}{h^2}ST^{m+1} + \frac{C}{h}RT^{m+1}$$
$$+ C\sum_{k=0}^{m}\Delta t\left(\left\|x^{k+1}-x_h^{k+1}\right\|^2 + \left\|y^{k+1}-y_h^{k+1}\right\|^2 + \left\|\left|x_u^{k+1}\right| - \left|x_{hu}^{k+1}\right|\right\|^2 + \frac{1}{h^5}RT^{k+1}\right).$$

Proof. From Lemma 4.6 after multiplying both sides by Δt and summing up from $k = 0$ to $k = m$ we get

$$\frac{c_0}{16}\sum_{k=0}^{m}\Delta t\left\|\frac{e_h^{k+1}-e_h^k}{\Delta t}\right\|^2 + \sum_{k=0}^{m}\left(\zeta^{k+1}-\zeta^k\right) \le C\sum_{k=0}^{m}\Delta t\left(h^2+\Delta t^2\right) + Ch^{-3}\sum_{k=0}^{m}\Delta t RT^{k+1}$$
$$+ C\sum_{k=0}^{m}\Delta t\left(\left\|y^{k+1}-y_h^{k+1}\right\|_{H^1}^2 + \left\|\left|x_u^{k+1}\right| - \left|x_{hu}^{k+1}\right|\right\|^2 + \left\|\tau^{k+1}-\tau_h^{k+1}\right\|^2\right),$$

where ζ^m is given by (4.39).
Recalling the initial condition $x_h^0 = I_h x^0$ and estimating $\zeta^0 \le Ch^2$ by (2.6) and (2.7) we may write

$$\frac{c_0}{16}\sum_{k=0}^{m}\Delta t\left\|\frac{e_h^{k+1}-e_h^k}{\Delta t}\right\|^2 + \zeta^{m+1} \le C\left(h^2+\Delta t^2\right) + Ch^{-3}\sum_{k=0}^{m}\Delta t RT^{k+1}$$
$$+ C\sum_{k=0}^{m}\Delta t\left(\left\|y^{k+1}-y_h^{k+1}\right\|_{H^1}^2 + \left\|\left|x_u^{k+1}\right| - \left|x_{hu}^{k+1}\right|\right\|^2 + \left\|\tau^{k+1}-\tau_h^{k+1}\right\|^2\right).$$

The next term we estimate from below

$$
\begin{aligned}
\zeta^{m+1} =& \frac{1}{2}\int_0^{2\pi} I_h\left[\left|I_h y^{m+1} - y_h^{m+1}\right|^2\right]\left|x_{hu}^{m+1}\right| - \frac{1}{4}\int_0^{2\pi}\left|y^{m+1}\right|^2\left|\tau^{m+1} - \tau_h^{m+1}\right|^2\left|x_{hu}^{m+1}\right| \\
&+ \int_0^{2\pi}\left(\left(\frac{\left|x_{hu}^{m+1}\right| - \left|x_u^{m+1}\right|}{\left|x_u^{m+1}\right|}\right)\left(\tau_h^{m+1} - \tau^{m+1}\right)\right. \\
&\qquad\left. + \frac{1}{2}\frac{\left|x_{hu}^{m+1}\right|}{\left|x_u^{m+1}\right|}\left|\tau_h^{m+1} - \tau^{m+1}\right|^2 \tau^{m+1}, y_u^{m+1}\right) \\
\geq& \frac{c_0}{8}\int_0^{2\pi}\left|I_h y^{m+1} - y_h^{m+1}\right|^2 - C\int_0^{2\pi}\left|\tau^{m+1} - \tau_h^{m+1}\right|^2 \\
&- C\int_0^{2\pi}\left|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right|\left|\tau^{m+1} - \tau_h^{m+1}\right|.
\end{aligned}
$$

Let us consider separately the first term on the right-hand side of the above inequality. Using Young's inequality (2.3) with $\varepsilon = 1$ we derive

$$
\begin{aligned}
\left|I_h y^{m+1} - y_h^{m+1}\right|^2 \geq& \left|I_h y^{m+1} - y^{m+1}\right|^2 + \left|y^{m+1} - y_h^{m+1}\right|^2 \\
&- 2\left|I_h y^{m+1} - y^{m+1}\right|\left|y^{m+1} - y_h^{m+1}\right| \\
\geq& -\left|I_h y^{m+1} - y^{m+1}\right|^2 + \frac{1}{2}\left|y^{m+1} - y_h^{m+1}\right|^2
\end{aligned}
$$

and with the help of (2.7), Cauchy-Schwarz and Young's inequalities we finally obtain

$$
\zeta^{m+1} \geq \frac{c_0}{16}\left\|y^{m+1} - y_h^{m+1}\right\|^2 - \delta\left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 - C_\delta\left(h^4 + \left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2\right).
$$

Taking into account the above result for ζ^{m+1} we deduce

$$
\begin{aligned}
&\frac{c_0}{16}\sum_{k=0}^m \Delta t\left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2 + \frac{c_0}{16}\left\|y^{m+1} - y_h^{m+1}\right\|^2 \leq C\left(h^2 + \Delta t^2\right) + Ch^{-3}\sum_{k=0}^m \Delta t RT^{k+1} \\
&+ \delta\left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 + C_\delta h^4 + C_\delta\left\|\tau^{m+1} - \tau_h^{m+1}\right\|^2 \\
&+ C\sum_{k=0}^m \Delta t\left(\left\|y^{k+1} - y_h^{k+1}\right\|_{H^1}^2 + \left\|\left|x_u^{k+1}\right| - \left|x_{hu}^{k+1}\right|\right\|^2 + \left\|\tau^{k+1} - \tau_h^{k+1}\right\|^2\right).
\end{aligned}
$$

Applying now Lemma 4.7 for the third term on the right-hand side of the above inequality one obtains

$$
\begin{aligned}
&\frac{c_0}{16}\sum_{k=0}^m \Delta t\left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2 + \frac{c_0}{16}\left\|y^{m+1} - y_h^{m+1}\right\|^2 \leq C\left(h^2 + \Delta t^2\right) + C_\delta\left(1 + C_\varepsilon\right)h^2 \\
&+ \left(\delta + \varepsilon C_\delta\right)\left\|\left|x_u^{m+1}\right| - \left|x_{hu}^{m+1}\right|\right\|^2 + \varepsilon C_\delta\left\|y^{m+1} - y_h^{m+1}\right\|^2 \\
&+ C_\delta C_\varepsilon\left\|x^{m+1} - x_h^{m+1}\right\|^2 + C_\delta C_\varepsilon h^{-1} RT^{m+1} + Ch^{-3}\sum_{k=0}^m \Delta t RT^{k+1} \\
&+ C\sum_{k=0}^m \Delta t\left(\left\|y^{k+1} - y_h^{k+1}\right\|_{H^1}^2 + \left\|\left|x_u^{k+1}\right| - \left|x_{hu}^{k+1}\right|\right\|^2 + \left\|\tau^{k+1} - \tau_h^{k+1}\right\|^2\right).
\end{aligned}
$$

At the same time Lemma 4.10 implies

$$\sum_{k=0}^{m} \Delta t \left\| y_u^{m+1} - y_{hu}^{m+1} \right\|^2 \le \left(\varepsilon + C \Delta t^2 h^{-4} \right) \sum_{k=0}^{m} \Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|^2 + C_\varepsilon h^{-5} \sum_{k=0}^{m} \Delta t R T^{k+1}$$
$$+ C_\varepsilon \sum_{k=0}^{m} \Delta t \left(h^2 + \Delta t^2 + \left\| x^{k+1} - x_h^{k+1} \right\|^2 + \left\| y^{k+1} - y_h^{k+1} \right\|^2 + \left\| |x_u^{k+1}| - |x_{hu}^{k+1}| \right\|^2 \right)$$

and yields

$$\frac{c_0}{16} \sum_{k=0}^{m} \Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|^2 + \frac{c_0}{16} \left\| y^{m+1} - y_h^{m+1} \right\|^2 \le C \left(1 + C_\varepsilon \right) \left(h^2 + \Delta t^2 \right) + C_\delta \left(1 + C_\varepsilon \right) h^2$$
$$+ \left(\delta + \varepsilon C_\delta \right) \left\| |x_u^{m+1}| - |x_{hu}^{m+1}| \right\|^2 + \varepsilon C_\delta \left\| y^{m+1} - y_h^{m+1} \right\|^2 + C_\delta C_\varepsilon \left\| x^{m+1} - x_h^{m+1} \right\|^2$$
$$+ C \left(\varepsilon + \frac{\Delta t^2}{h^4} \right) \sum_{k=0}^{m} \Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|^2 + C_\delta C_\varepsilon h^{-1} R T^{m+1} + C \left(h^{-3} + C_\varepsilon h^{-5} \right) \sum_{k=0}^{m} \Delta t R T^{k+1}$$
$$+ C \sum_{k=0}^{m} \Delta t \left(\left\| y^{k+1} - y_h^{k+1} \right\|^2 + \left\| |x_u^{k+1}| - |x_{hu}^{k+1}| \right\|^2 + \left\| \tau^{k+1} - \tau_h^{k+1} \right\|^2 \right)$$
$$+ C C_\varepsilon \sum_{k=0}^{m} \Delta t \left(\left\| x^{k+1} - x_h^{k+1} \right\|^2 + \left\| y^{k+1} - y_h^{k+1} \right\|^2 + \left\| |x_u^{k+1}| - |x_{hu}^{k+1}| \right\|^2 \right).$$

As the next step we make use of Lemma 4.12 for the third term on the right-hand side. The previous inequality becomes

$$\frac{c_0}{16} \sum_{k=0}^{m} \Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|^2 + \frac{c_0}{16} \left\| y^{m+1} - y_h^{m+1} \right\|^2 \le C \left(1 + \delta + \varepsilon C_\delta + C_\varepsilon \right) \left(h^2 + \Delta t^2 \right)$$
$$+ C_\delta \left(1 + C_\varepsilon \right) h^2 + \varepsilon C_\delta \left\| y^{m+1} - y_h^{m+1} \right\|^2 + C_\delta C_\varepsilon \left\| x^{m+1} - x_h^{m+1} \right\|^2$$
$$+ \left(\delta + \varepsilon \left(C + C_\delta \right) + C \frac{\Delta t^2}{h^4} \right) \sum_{k=0}^{m} \Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|^2 + \left(\delta + \varepsilon C_\delta \right) C \Delta t h^{-2} S T^{m+1}$$
$$+ \left(\delta + \varepsilon C_\delta + C_\delta C_\varepsilon \right) h^{-1} R T^{m+1} + C \left(h^{-3} + h^{-5} \left(C_\varepsilon + \delta + \varepsilon C_\delta \right) \right) \sum_{k=0}^{m} \Delta t R T^{k+1}$$
$$+ C \sum_{k=0}^{m} \Delta t \left(\left\| y^{k+1} - y_h^{k+1} \right\|^2 + \left\| \tau^{k+1} - \tau_h^{k+1} \right\|^2 + \left(1 + C_\varepsilon \right) \left\| |x_u^{k+1}| - |x_{hu}^{k+1}| \right\|^2 \right)$$
$$+ \left(\delta + \varepsilon C_\delta + C C_\varepsilon \right) \sum_{k=0}^{m} \Delta t \left(\left\| x^{k+1} - x_h^{k+1} \right\|^2 + \left\| y^{k+1} - y_h^{k+1} \right\|^2 \right).$$

Choosing first δ and then ε sufficiently small we let ω to be equal $\frac{1}{8}\sqrt{\frac{c_0}{C}}$.

The resulting inequality reads then as follows

$$\frac{c_0}{32}\sum_{k=0}^{m}\Delta t\left\|\frac{e_h^{k+1}-e_h^k}{\Delta t}\right\|^2+\frac{c_0}{32}\left\|y^{m+1}-y_h^{m+1}\right\|^2\leq C\left(h^2+\Delta t^2\right)$$

$$+C\left\|x^{m+1}-x_h^{m+1}\right\|^2+Ch^{-1}RT^{m+1}+Ch^{-5}\sum_{k=0}^{m}\Delta tRT^{k+1}+C\Delta th^{-2}ST^{m+1}$$

$$+C\sum_{k=0}^{m}\Delta t\left(\left\|x^{k+1}-x_h^{k+1}\right\|^2+\left\|y^{k+1}-y_h^{k+1}\right\|^2+\left\|\left|x_u^{k+1}\right|-\left|x_{hu}^{k+1}\right|\right\|^2+\left\|\tau^{k+1}-\tau_h^{k+1}\right\|^2\right).$$

In order to estimate the second term on the right-hand side of the above inequality we observe

$$\left\|x^{m+1}-x_h^{m+1}\right\|^2-\left\|x^0-x_h^0\right\|^2=\sum_{k=0}^{m}\int_0^{2\pi}\left|x^{k+1}-x_h^{k+1}\right|^2-\left|x^k-x_h^k\right|^2$$

$$=\sum_{k=0}^{m}\int_0^{2\pi}\left(x^{k+1}-x_h^{k+1}-\left(x^k-x_h^k\right),\left(x^{k+1}-x_h^{k+1}\right)+\left(x^k-x_h^k\right)\right)$$

$$=\sum_{k=0}^{m}\int_0^{2\pi}\Delta t\left(\frac{e_h^{k+1}-e_h^k}{\Delta t},\left(x^{k+1}-x_h^{k+1}\right)+\left(x^k-x_h^k\right)\right)$$

$$-\sum_{k=0}^{m}\int_0^{2\pi}\Delta t\left(I_h\left[\frac{x^{k+1}-x^k}{\Delta t}\right]-\frac{x^{k+1}-x^k}{\Delta t},\left(x^{k+1}-x_h^{k+1}\right)+\left(x^k-x_h^k\right)\right).$$

Recalling the initial condition, Cauchy's and Young's inequalities we obtain for $\delta>0$

$$\left\|x^{m+1}-x_h^{m+1}\right\|^2\leq\delta\sum_{k=0}^{m}\Delta t\left\|\frac{e_h^{k+1}-e_h^k}{\Delta t}\right\|^2+C_\delta h^2+C_\delta\sum_{k=0}^{m}\Delta t\left\|x^{k+1}-x_h^{k+1}\right\|^2. \tag{5.1}$$

The claim follows by choosing δ sufficiently small and using Lemma 4.7 with $\varepsilon=1$. □

We are now in position to complete the proof of Theorem 4.2. For convenience, we formulate the following theorem.

Theorem 5.2 (Function ρ^{m+1}). *Let (x_h^k,y_h^k), $0\leq k\leq m$ be given, such that (4.8) holds. Then there exist $h_0>0$, $\Delta t_0>0$, $\hat{c}>0$, $\gamma>0$ and $0<\xi\leq1$, such that if the function ρ^m, given by (4.7), satisfies*

$$\rho^m\leq\hat{c}e^{\gamma m\Delta t}\left(h^2+\Delta t^2\right), \tag{5.2}$$

then

$$\rho^{m+1}\leq\hat{c}e^{\gamma(m+1)\Delta t}\left(h^2+\Delta t^2\right) \tag{5.3}$$

for all $0<h\leq h_0$, $0<\Delta t\leq\Delta t_0$, provided that $\Delta t\leq\xi h^{\frac{7}{2}}$. The constants depend only on the norms of the continuous solution and final time T.

Proof. Let us now show that (5.3) holds. Recalling the definition (4.7) and using (5.1) with $\delta = 1$ and Lemma 4.12 we arrive at

$$\begin{aligned}\rho^{m+1} =& \left\|x^{m+1} - x_h^{m+1}\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2 + \left\||x_u^{m+1}| - |x_{hu}^{m+1}|\right\|^2 + \sum_{k=0}^{m} \Delta t \left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2 \\ \leq& C\left(h^2 + \Delta t^2\right) + \left\|y^{m+1} - y_h^{m+1}\right\|^2 + C\frac{\Delta t}{h^2} ST^{m+1} + \frac{C}{h} RT^{m+1} + Ch^{-5}\sum_{k=0}^{m} \Delta t RT^{k+1} \\ &+ C\sum_{k=0}^{m} \Delta t \left(\left\|x^{k+1} - x_h^{k+1}\right\|^2 + \left\|y^{k+1} - y_h^{k+1}\right\|^2 + \left(1 + \frac{\Delta t^2}{h^4}\right)\left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2\right).\end{aligned}$$

In view of Lemma 5.1 we deduce

$$\begin{aligned}\rho^{m+1} \leq& C\left(h^2 + \Delta t^2\right) + C\sum_{k=0}^{m} \Delta t \left(\left\|x^{k+1} - x_h^{k+1}\right\|^2 + \left\|y^{k+1} - y_h^{k+1}\right\|^2 + \left\||x_u^{k+1}| - |x_{hu}^{k+1}|\right\|^2\right) \\ &+ C\frac{\Delta t}{h^2} ST^{m+1} + \frac{C}{h} RT^{m+1} + \frac{C}{h^5}\sum_{k=0}^{m} \Delta t RT^{k+1} + C\frac{\Delta t^2}{h^4}\sum_{k=0}^{m} \Delta t \left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2.\end{aligned}$$

Next, we split the second term on the right-hand side of the above inequality into a sum of the last summand $\Delta t\left(\left\|x^{m+1} - x_h^{m+1}\right\|^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2 + \left\||x_u^{m+1}| - |x_{hu}^{m+1}|\right\|^2\right)$ and the remaining sum till $m-1$. Further, smallness of Δt allows us to move the above term to the left-hand side. Here, without lost of generality we may assume $\Delta t \leq \frac{1}{2}$. Now, discrete Gronwall's Lemma applies and yields

$$\begin{aligned}\rho^{m+1} \leq& C\left(h^2 + \Delta t^2\right) + C\Delta t h^{-2} ST^{m+1} + C\left(h^{-1} + \Delta t h^{-5}\right) RT^{m+1} \\ &+ Ch^{-5}\sum_{k=0}^{m-1} \Delta t RT^{k+1} + C\Delta t^2 h^{-4}\sum_{k=0}^{m} \Delta t \left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2.\end{aligned} \tag{5.4}$$

It remains to examine the second, third and fourth terms on the right-hand side of the above inequality. In view of (4.50) we obtain for ST^{m+1} given by (4.115)

$$ST^{m+1} = \sum_{k=0}^{m} \left\|\tau_h^{k+1} - \tau_h^k\right\|^2 \leq C\Delta t h^{-2}\sum_{k=0}^{m} \Delta t \left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2 + C\Delta t.$$

Recalling the definition (4.17) of RT^{m+1} and taking into account (3.25) and the induction hypothesis (4.9) we deduce

$$\begin{aligned}RT^{m+1} =& \left(\sum_{k=0}^{m-1} \left\|\tau_h^{k+1} - \tau_h^k\right\|^2 + \left\|\tau_h^{m+1} - \tau_h^m\right\|^2\right)\sum_{k=0}^{m} \left\|\tau_h^{k+1} - \tau_h^k\right\|^2 \\ \leq& C\left(\Delta t h^{-2}\sum_{k=0}^{m-1} \Delta t \left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2 + \Delta t + \left\|x_{hu}^{m+1} - x_{hu}^m\right\|^2\right)\sum_{k=0}^{m} \left\|\tau_h^{k+1} - \tau_h^k\right\|^2 \\ \leq& C\left(\Delta t h^{-2}\hat{c}e^{\gamma m\Delta t}\left(h^2 + \Delta t^2\right) + \Delta t + \Delta t h^{-2}\right)\left(\frac{\Delta t}{h^2}\sum_{k=0}^{m} \Delta t \left\|\frac{e_h^{k+1} - e_h^k}{\Delta t}\right\|^2 + \Delta t\right).\end{aligned}$$

Analogously, for the fourth term on the right-hand side of (5.4) we have

$$\begin{aligned}\sum_{k=0}^{m-1}\Delta t\left(\sum_{l=0}^{k}\left\|\tau_h^{l+1}-\tau_h^{l}\right\|^2\right)^2 \leq & C\sum_{k=0}^{m}\Delta t\left(\Delta t h^{-2}\hat{c}e^{\gamma(k+1)\Delta t}\left(h^2+\Delta t^2\right)+\Delta t\right)^2\\ \leq & C\Delta t^2 + C\hat{c}^2\Delta t^2 h^{-4}\left(h^2+\Delta t^2\right)^2\sum_{k=0}^{m-1}\Delta t e^{2\gamma(k+1)\Delta t}.\end{aligned}$$

We note that here the induction hypothesis applies, since the first sum goes till $m-1$. Exploiting the formula for the sum of the first m terms of a geometric series we obtain

$$\Delta t\sum_{k=0}^{m-1}e^{2\gamma(k+1)\Delta t} = \Delta t e^{2\gamma\Delta t}\frac{e^{2\gamma m\Delta t}-1}{e^{2\gamma\Delta t}-1} \leq \Delta t e^{2\gamma\Delta t}\frac{e^{2\gamma m\Delta t}}{e^{2\gamma\Delta t}-1} \leq \Delta t\frac{e^{2\gamma(m+1)\Delta t}}{2\gamma\Delta t} = \frac{e^{2\gamma(m+1)\Delta t}}{2\gamma}.$$

Here, we used the Taylor expansion of the exponential function and $e^{2\gamma\Delta t}-1 \geq 2\gamma\Delta t > 0$. Hence,

$$\sum_{k=0}^{m-1}\Delta t RT^{k+1} \leq C\Delta t^2 + C\frac{\hat{c}^2}{2\gamma}e^{2\gamma(m+1)\Delta t}\Delta t^2 h^{-4}\left(h^2+\Delta t^2\right)^2.$$

In view of the above calculations and after rearranging the coefficients, (5.4) takes the form

$$\begin{aligned}\rho^{m+1} \leq & C\left(h^2+\Delta t^2\right) + C\frac{\hat{c}^2}{\gamma}e^{2\gamma(m+1)\Delta t}\Delta t^2 h^{-9}\left(h^2+\Delta t^2\right)^2 + C\Delta t^2\left(h^{-5}+h^{-2}\right)\\ & + C\left(h^{-1}+\Delta t h^{-5}\right)\left(\hat{c}e^{\gamma m\Delta t}\left(\Delta t^2+\Delta t^4 h^{-2}\right)+\Delta t^2 h^{-2}\right)\\ & + C\left(\left(\frac{1}{h}+\frac{\Delta t}{h^5}\right)\left(\hat{c}e^{\gamma m\Delta t}\left(\frac{\Delta t^2}{h^2}+\frac{\Delta t^4}{h^4}\right)+\frac{\Delta t^2}{h^4}\right)+\frac{\Delta t^2}{h^4}\right)\sum_{k=0}^{m}\Delta t\left\|\frac{e_h^{k+1}-e_h^k}{\Delta t}\right\|^2\\ = & I + \ldots + V.\end{aligned}$$

Let us examine the above terms in more detail. We shall omit in our further consideration the generic constant C, concentrating only on the crucial terms, and using it again when collecting all the results together. The second term in view of the relation $M\Delta t = T$ and under the condition $\Delta t \leq \xi h^{\frac{7}{2}}$ can be estimated as

$$\begin{aligned}II = & \frac{\hat{c}^2}{\gamma}e^{2\gamma(m+1)\Delta t}\Delta t^2 h^{-9}\left(h^2+\Delta t^2\right)^2 \leq \frac{\hat{c}}{\gamma}e^{\gamma(m+1)\Delta t}\left(h^2+\Delta t^2\right)\hat{c}e^{\gamma M\Delta t}\left(\Delta t^2 h^{-7}+\Delta t^4 h^{-9}\right)\\ \leq & \frac{\hat{c}}{\gamma}e^{\gamma(m+1)\Delta t}\left(h^2+\Delta t^2\right)\hat{c}e^{\gamma T}\left(\xi^2+\xi^2\Delta t^2 h^{-2}\right).\end{aligned}$$

Without lost of generality we may assume $\Delta t^2 h^{-2} \leq 1$. From what follows

$$II \leq \frac{\hat{c}}{\gamma}e^{\gamma(m+1)\Delta t}\left(h^2+\Delta t^2\right)2\xi^2\hat{c}e^{\gamma T} \leq \frac{\hat{c}}{\gamma}e^{\gamma(m+1)\Delta t}\left(h^2+\Delta t^2\right),$$

if the following condition is satisfied

$$2\xi^2 \hat{c} e^{\gamma T} \leq 1. \tag{5.5}$$

Here and after we shall assume that $\Delta t, h \leq 1$. On the other hand, condition $\Delta t \leq \xi h^{\frac{7}{2}}$ along with $0 < \xi \leq 1$ and Young's inequality implies for the third term

$$III = \Delta t^2 \left(h^{-5} + h^{-2}\right) \leq \xi^2 h^2 + \xi \Delta t h^{\frac{3}{2}} \leq h^2 + \Delta t h^{\frac{3}{2}} \leq C\left(h^2 + \Delta t^2\right).$$

Furthermore, from (5.5) and Young's inequality we derive

$$\begin{aligned} IV =& \hat{c} e^{\gamma m \Delta t} \left(\Delta t^2 h^{-1} + \Delta t^4 h^{-3} + \Delta t^3 h^{-5} + \Delta t^5 h^{-7}\right) + \Delta t^2 h^{-3} + \Delta t^3 h^{-7} \\ \leq& \hat{c} e^{\gamma T} \left(\xi^2 h^6 + \xi^2 \Delta t^2 h^4 + \xi^2 \Delta t h^2 + \xi^3 \Delta t^2 h^{\frac{7}{2}}\right) + \xi^2 h^4 + \xi^3 h^{\frac{7}{2}} \\ \leq& 2\xi^2 \hat{c} e^{\gamma T} \left(h^4 + \Delta t^2\right) + h^4 + h^{\frac{7}{2}} \leq C\left(h^2 + \Delta t^2\right). \end{aligned}$$

Combining the coefficients in front of the fifth term we analogously obtain

$$\begin{aligned} & \hat{c} e^{\gamma m \Delta t} \left(\Delta t^2 h^{-3} + \Delta t^4 h^{-5} + \Delta t^3 h^{-7} + \Delta t^5 h^{-9}\right) + \Delta t^2 h^{-5} + \Delta t^3 h^{-9} + \Delta t^2 h^{-4} \\ & \leq \hat{c} e^{\gamma T} \left(\xi^2 h^4 + \xi^2 \Delta t^2 h^2 + \xi^2 \Delta t + \xi^3 \Delta t^2 h^{\frac{3}{2}}\right) + \xi^2 h^2 + \xi^3 h^{\frac{3}{2}} + \xi^2 h^3 \\ & \leq 2\xi^2 \hat{c} e^{\gamma T} \left(h^3 + \Delta t\right) + 3\xi^2 h^{\frac{3}{2}} \leq h^3 + \Delta t + 3\xi^2 h^{\frac{3}{2}} \leq 5h^{\frac{3}{2}}. \end{aligned}$$

Let us collect the above results together

$$\begin{aligned} \rho^{m+1} \leq& C\left(h^2 + \Delta t^2\right) + C\frac{\hat{c}}{\gamma} e^{\gamma(m+1)\Delta t}\left(h^2 + \Delta t^2\right) + 5Ch^{\frac{3}{2}} \sum_{k=0}^{m} \Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|^2 \\ \leq& C\left(h^2 + \Delta t^2\right) + C\frac{\hat{c}}{\gamma} e^{\gamma(m+1)\Delta t}\left(h^2 + \Delta t^2\right) + 5Ch^{\frac{3}{2}} \rho^{m+1}. \end{aligned}$$

Here we used the definition of the function ρ^{m+1}. Further, provided h_0 is sufficiently small, the above inequality with a slightly different constant C turns into

$$\begin{aligned} \rho^{m+1} \leq& C\left(h^2 + \Delta t^2\right) + C\frac{\hat{c}}{\gamma} e^{\gamma(m+1)\Delta t}\left(h^2 + \Delta t^2\right) \\ \leq& C\left(1 + \frac{\hat{c}}{\gamma}\right) e^{\gamma(m+1)\Delta t}\left(h^2 + \Delta t^2\right) \overset{?}{\leq} \hat{c} e^{\gamma(m+1)\Delta t}\left(h^2 + \Delta t^2\right). \end{aligned}$$

In order to satisfy the above inequality, we choose the constants in the following way

$$\begin{aligned} C\left(1 + \frac{\hat{c}}{\gamma}\right) &\leq \hat{c}, \\ C &\leq \hat{c}\left(1 - \frac{C}{\gamma}\right). \end{aligned}$$

Let $\dfrac{C}{\gamma} = \dfrac{1}{2}$, then $\gamma = 2C$ and $\hat{c} = 2C$. Since the constants γ and $\hat{c}$ already fixed, we can now make a choice for ξ. Let $\xi = \min\left\{1, \dfrac{1}{2\sqrt{Ce^{2CT}}}\right\}$, so that the condition (5.5) is satisfied. And the assertion of the lemma follows.

□

5.2 A posteriori estimates

From Lemma 4.10 with $\varepsilon = 1$ and Theorem 4.2 under the made assumptions on $\hat{c}, \gamma$ and Δt with a generic constant C, which may vary from line to line, follows

$$\sum_{k=0}^{m} \Delta t \left\| y_u^{m+1} - y_{hu}^{m+1} \right\|^2 \leq \left(1 + C\Delta t^2 h^{-4}\right) \sum_{k=0}^{m} \Delta t \left\| \frac{e_h^{k+1} - e_h^k}{\Delta t} \right\|^2 + Ch^{-5} \sum_{k=0}^{m} \Delta t RT^{k+1}$$
$$+ C\left(h^2 + \Delta t^2\right) + C \sum_{k=0}^{m} \Delta t \left(\left\| x^{k+1} - x_h^{k+1} \right\|^2 + \left\| y^{k+1} - y_h^{k+1} \right\|^2 + \left\| \left| x_u^{k+1} \right| - \left| x_{hu}^{k+1} \right| \right\|^2 \right)$$
$$\leq C \left(1 + \hat{c} e^{\gamma (m+1) \Delta t} \left(1 + \Delta t + \Delta t^2 h^{-4} + \frac{1}{\gamma} \right) \right) \left(h^2 + \Delta t^2\right) \leq C\left(h^2 + \Delta t^2\right).$$

Thus, we obtain

$$\underset{m=0,\ldots,M}{max} \left\| x^m - x_h^m \right\|_{H^1}^2 + \left\| y^m - y_h^m \right\|^2$$
$$+ \sum_{m=0}^{M-1} \Delta t \left(\left\| y_u^{m+1} - y_{hu}^{m+1} \right\|^2 + \left\| \frac{e^{m+1} - e^m}{\Delta t} \right\|^2 \right) \leq C\left(h^2 + \Delta t^2\right).$$

Here, we have used the relation (4.27)

$$x_u^{m+1} - x_{hu}^{m+1} = \left(\tau^{m+1} - \tau_h^{m+1}\right) \left| x_u^{m+1} \right| + \tau_h^{m+1} \left(\left| x_u^{m+1} \right| - \left| x_{hu}^{m+1} \right| \right)$$

and Lemma 4.7.

The smoothness of the continuous solution together with the above inequality and the inverse estimates (2.11), (2.12) allows us to write

$$\left\| x_u^{m+1} - x_{hu}^{m+1} \right\|_{L^\infty}, \left\| y^{m+1} - y_h^{m+1} \right\|_{L^\infty} \leq C \left(\sqrt{h} + \frac{\Delta t}{\sqrt{h}} \right) \leq C\sqrt{h},$$
$$\sum_{k=0}^{m} \Delta t \left\| y_u^{m+1} - y_{hu}^{m+1} \right\|_{L^\infty}^2 \leq Ch.$$

Observing further (4.1) we achieve

$$\left| x_{hu}^{m+1} \right| \geq \left| x_u^{m+1} \right| - \left\| x_u^{m+1} - x_{hu}^{m+1} \right\|_{L^\infty} \geq c_0 - C\sqrt{h} > \frac{3}{4} c_0,$$
$$\left| x_{hu}^{m+1} \right| \leq \left\| x_u^{m+1} - x_{hu}^{m+1} \right\|_{L^\infty} + \left| x_u^{m+1} \right| \leq C\sqrt{h} + C_0 < \frac{3}{2} C_0.$$

Analogously, we obtain

$$\sum_{k=0}^{m} \Delta t \left\| y_{hu}^{k+1} \right\|_{L^\infty}^2 \leq \sum_{k=0}^{m} \Delta t \left\| y_u^{k+1} \right\|_{L^\infty}^2 + \sum_{k=0}^{m} \Delta t \left\| y_u^{k+1} - y_{hu}^{k+1} \right\|_{L^\infty}^2 \leq C_0 + Ch < \frac{3}{2} C_0.$$

To sum up, we conclude that being given a solution (x_h^m, y_h^m) at the previous time step, we are able to find the solution $\left(x_h^{m+1}, y_h^{m+1}\right)$ at the next time step. Performing these steps repeatedly allows to find all the solutions (x_h^m, y_h^m) for $m = 1, ..., M$. Therefore, the proof of Theorem 4.1 together with the proof of the error bounds is completed.

Chapter 6

Summary

The major goal of the thesis was to carry out an error analysis of a fully discrete numerical scheme to approximate the elastic flow of curves.

An L^2-gradient flow of the modified elastic energy functional E_λ defined by (1.1) leads to the evolution equation (1.2). This equation is of fourth-order and represents a parabolic system of partial differential equations. To solve the problem numerically, we split the fourth-order problem into two coupled second order equations. The proposed fully discrete numerical scheme (1.11)-(1.12) is based on the weak formulation of the continuous and continuous-in-time semidiscrete problems. The chosen finite element set consists of the continuous piecewise linear functions. In order to prove the existence of the unique solution to (1.11)-(1.12), we formulated a suitable constrained minimization problem to be solved in each time step. From the existence of the minimizer followed the existence of the discrete solution at the next time step. Under certain restrictions ($\Delta t \leq \mu h^5$) it was possible to obtain the unique solution of the system satisfying specified bounds.

To begin the error analysis, we formulated an induction-type argument with a hypothesis

$$\begin{aligned} &\left\|x^{m+1} - x_h^{m+1}\right\|_{H^1}^2 + \left\|y^{m+1} - y_h^{m+1}\right\|^2 \\ &\qquad + \sum_{k=0}^{m} \Delta t \left(\left\|y_u^{k+1} - y_{hu}^{k+1}\right\|^2 + \left\| \frac{e^{k+1} - e^k}{\Delta t} \right\|^2 \right) \leq C\left(h^2 + \Delta t^2\right). \end{aligned} \tag{6.1}$$

and showed that (6.1) is true for $m = 0$. Assuming the induction hypothesis for some m and applying the existence result we obtained a priori bounds on the discrete solution at the next time step, which were required to carry out the error analysis. To estimate the error between continuous and discrete solutions, we inserted test functions into the equations of the scheme and derived the estimates of certain norms. Due to the degeneracy of the equation (1.2) in tangential direction, we treated the space derivative of the position vector in several lemmas for the direction and length, respectively. A discrete Gronwall's argument completed this proof. After the error analysis was closed, we proved

the induction step and improved a priori bounds on the discrete solution.
Our main result is a derivation of the following error bounds

$$\max_{m=0,\dots,M} \|x^m - x_h^m\|_{H^1}^2 + \|y^m - y_h^m\|^2 + \sum_{m=0}^{M-1} \Delta t \left(\left\|y_u^{m+1} - y_{hu}^{m+1}\right\|^2 + \left\| \frac{e^{m+1} - e^m}{\Delta t} \right\|^2 \right) \leq C\left(h^2 + \Delta t^2\right),$$

where $e^m = x^m - x_h^m$, provided that $\Delta t \leq \mu h^5$. The constants depend only on the certain norms of the continuous solution and final time $T = M\Delta t$.

In the following, we would like to mention some interesting facts and questions, related to the subject of the thesis.

- To ensure the unique solvability of the fully discrete problem, one has to impose the restriction $\Delta t \leq \varepsilon h^3$ (see, Remark 3.4).
- The induction step (see, Theorem 5.2), which was used to prove error bounds, was carried out under the assumption $\Delta t \leq \xi h^{\frac{7}{2}}$.
- The condition $\Delta t \leq \mu h^5$ was crucial for the error analysis, since it was used to control the curvature vector at the next time step.

  In this connection, the following questions are of interest:

  - Is it possible to prove the induction step under the milder condition $\Delta t \leq \varepsilon h^3$?
  - Is there any way to improve the condition $\Delta t \leq \mu h^5$ or gain uniform control on the curvature vector in a different way?
- Is there a fully discrete numerical scheme to approximate the solution of (1.2), which is linear at each time step and can be analyzed?

Appendix A

In this chapter, we present 4 auxiliary lemmas, in which we derive the proof for the results formulated in Lemma 4.11.

Lemma A.1. *The first scalar product on the right-hand side of (4.105) can be written as*

$$\begin{aligned}\left(\tau_j^{m+1}, \frac{x_j^{m+1}-x_j^m}{\Delta t}\right) = &\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left(\left|y_{j+1}^{m+1}\right|^2 - \left|y_j^{m+1}\right|^2\right) \\ &- \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_{j+1}^{m+1} - y_j^{m+1}\right|^2 \\ &+ \frac{1}{4}\frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_{j+1}^{m+1}\left|y_{j+1}^{m+1}\right|^2 + \alpha_j^{m+1}\left|y_j^{m+1}\right|^2\right) \\ &- \frac{1}{2}\frac{1}{\alpha_j^m}\left(J_{j+1}^{m+1} - J_j^{m+1}\right) + \frac{1}{4}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2 J_{j+1}^{m+1} \\ &- \frac{\lambda}{2}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2 + D_j^{m+1},\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}D_j^{m+1} = &\frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1} - y_j^{m+1}, R_j^{m+1}\right) \\ &+ \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(RP_{j+1}^{m+1} + RM_j^{m+1}\right) \\ &+ \frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^m\right)\left(\tau_{j+1}^{m+1} - \tau_{j+1}^m, \alpha_j^{m+1}y_j^{m+1}\right) \\ &- \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^m - \tau_{j+1}^{m+1}\right) \\ &- \frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(\alpha_j^{m+1}RM_j^{m+1} + \left(\tau_{j+1}^m, R_j^{m+1}\right)\right)\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^m\right)\end{aligned} \tag{A.2}$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1}-y_j^{m+1},\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right)\left|\tau_{j+1}^{m+1}-\tau_j^{m+1}\right|^2\\
&-\frac{1}{4}\frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1}-y_j^{m+1},\tau_{j+1}^{m}\right)\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|^2\left|\tau_{j+1}^{m+1}-\tau_j^{m+1}\right|^2\\
&+\frac{1}{2}\frac{1}{\alpha_j^m}RS_j^{m+1}J_{j+1}^{m+1}-\lambda\frac{1}{\alpha_j^m}RS_j^{m+1}
\end{aligned}
$$

and

$$
\begin{aligned}
R_j^{m+1}&=-\frac{1}{2}\sum_{k=0}^{m}\left(\tau_{j+1}^{k}\left|\tau_{j+1}^{k+1}-\tau_{j+1}^{k}\right|^2-\tau_j^k\left|\tau_j^{k+1}-\tau_j^k\right|^2\right),\\
RS_j^{m+1}&=\alpha_j^{m+1}\left(y_j^{m+1},R_j^{m+1}\right)+\frac{1}{2}\left|R_j^{m+1}\right|^2,\\
RM_j^{m+1}&=\left(y_j^{m+1},R_j^{m+1}\right)+\frac{1}{2}\frac{1}{\alpha_j^{m+1}}\left|R_j^{m+1}\right|^2-\frac{1}{\alpha_j^{m+1}}\left(\tau_{j+1}^{m+1},R_j^{m+1}\right),\\
RP_j^{m+1}&=\left(y_j^{m+1},R_j^{m+1}\right)+\frac{1}{2}\frac{1}{\alpha_j^{m+1}}\left|R_j^{m+1}\right|^2+\frac{1}{\alpha_j^{m+1}}\left(\tau_j^{m+1},R_j^{m+1}\right).
\end{aligned}
\tag{A.3}
$$

Proof. Taking a scalar product between τ_j^{m+1} and (4.78) we derive

$$
\begin{aligned}
\alpha_j^m\left(\tau_j^{m+1},\frac{x_j^{m+1}-x_j^m}{\Delta t}\right)&=-\frac{\left(\tau_j^{m+1},P_{j+1}^m\left(y_{j+1}^{m+1}-y_j^{m+1}\right)\right)}{q_{j+1}^{m+1}}+\frac{\left(\tau_j^{m+1},P_j^m\left(y_j^{m+1}-y_{j-1}^{m+1}\right)\right)}{q_j^{m+1}}\\
&-\frac{1}{4}\left(\left(\left|y_j^{m+1}\right|^2+\left|y_{j+1}^{m+1}\right|^2\right)\left(\tau_j^{m+1},\tau_{j+1}^{m+1}\right)-\left(\left|y_{j-1}^{m+1}\right|^2+\left|y_j^{m+1}\right|^2\right)\left(\tau_j^{m+1},\tau_j^{m+1}\right)\right)\\
&+\frac{\left(P_{j+1}^m\left(y_{j+1}^{m+1}-y_j^{m+1}\right),\tau_{j+1}^{m+1}\right)}{q_{j+1}^{m+1}}\left(\tau_{j+1}^{m+1},\tau_j^{m+1}\right)\\
&-\frac{\left(P_j^m\left(y_j^{m+1}-y_{j-1}^{m+1}\right),\tau_j^{m+1}\right)}{q_j^{m+1}}\left(\tau_j^{m+1},\tau_j^{m+1}\right)+\lambda\left(\tau_{j+1}^{m+1}-\tau_j^{m+1},\tau_j^{m+1}\right).
\end{aligned}
$$

We observe that sum of the second and fifth terms on the right-hand side results in zero. After introducing the following abbreviation

$$
J_j^{m+1}=\frac{1}{2}\left(\left|y_{j-1}^{m+1}\right|^2+\left|y_j^{m+1}\right|^2\right)
\tag{A.4}
$$

the above equation takes the form

$$
\begin{aligned}
&\alpha_j^m\left(\tau_j^{m+1},\frac{x_j^{m+1}-x_j^m}{\Delta t}\right)\\
&=-\frac{\left(P_{j+1}^m\left(y_{j+1}^{m+1}-y_j^{m+1}\right),\tau_j^{m+1}\right)}{q_{j+1}^{m+1}}-\frac{1}{2}\left(J_{j+1}^{m+1}\left(\tau_j^{m+1},\tau_{j+1}^{m+1}\right)-J_j^{m+1}\right)\\
&+\frac{\left(P_{j+1}^m\left(y_{j+1}^{m+1}-y_j^{m+1}\right),\tau_{j+1}^{m+1}\right)}{q_{j+1}^{m+1}}\left(\tau_{j+1}^{m+1},\tau_j^{m+1}\right)+\lambda\left(\tau_{j+1}^{m+1}-\tau_j^{m+1},\tau_j^{m+1}\right).
\end{aligned}
\tag{A.5}
$$

Recalling the symmetry property of the projection matrix and identity (3.21) we deduce for the first and third terms on the right-hand side of (A.5)

$$\begin{aligned}
&-\frac{\left(P^m_{j+1}\tau^{m+1}_j, y^{m+1}_{j+1}-y^{m+1}_j\right)}{q^{m+1}_{j+1}}+\frac{\left(P^m_{j+1}\tau^{m+1}_{j+1}, y^{m+1}_{j+1}-y^{m+1}_j\right)}{q^{m+1}_{j+1}}\left(\tau^{m+1}_{j+1},\tau^{m+1}_j\right)\\
&\quad=\frac{\left(P^m_{j+1}\left(\tau^{m+1}_{j+1}-\tau^{m+1}_j\right), y^{m+1}_{j+1}-y^{m+1}_j\right)}{q^{m+1}_{j+1}}\\
&\quad-\frac{1}{2}\frac{\left(P^m_{j+1}\tau^{m+1}_{j+1}, y^{m+1}_{j+1}-y^{m+1}_j\right)}{q^{m+1}_{j+1}}\left|\tau^{m+1}_{j+1}-\tau^{m+1}_j\right|^2.
\end{aligned}\tag{A.6}$$

From the definition of the projection matrix (1.6) and relation (3.21) we infer

$$P^m_{j+1}\tau^{m+1}_{j+1}=\tau^{m+1}_{j+1}-\left(\tau^m_{j+1},\tau^{m+1}_{j+1}\right)\tau^m_{j+1}=\tau^{m+1}_{j+1}-\tau^m_{j+1}+\frac{1}{2}\left|\tau^{m+1}_{j+1}-\tau^m_{j+1}\right|^2\tau^m_{j+1}.\tag{A.7}$$

Application of (4.79) and (4.85) leads to

$$\begin{aligned}
&P^m_{j+1}\left(\tau^{m+1}_{j+1}-\tau^{m+1}_j\right)=\left(\tau^{m+1}_{j+1}-\tau^{m+1}_j\right)-\left(\tau^m_{j+1},\tau^{m+1}_{j+1}-\tau^{m+1}_j\right)\tau^m_{j+1}\\
&\quad=\alpha^{m+1}_j y^{m+1}_j+R^{m+1}_j-\left(\tau^m_{j+1},\alpha^{m+1}_j y^{m+1}_j+R^{m+1}_j\right)\tau^m_{j+1}\\
&\quad=\alpha^{m+1}_j y^{m+1}_j+R^{m+1}_j-\left(\tau^m_{j+1}-\tau^{m+1}_{j+1},\alpha^{m+1}_j y^{m+1}_j\right)\tau^m_{j+1}\\
&\quad-\alpha^{m+1}_j\left(\tau^{m+1}_{j+1},y^{m+1}_j\right)\tau^m_{j+1}-\left(\tau^m_{j+1},R^{m+1}_j\right)\tau^m_{j+1}\\
&\quad=\alpha^{m+1}_j y^{m+1}_j+R^{m+1}_j+\left(\tau^{m+1}_{j+1}-\tau^m_{j+1},\alpha^{m+1}_j y^{m+1}_j\right)\tau^m_{j+1}\\
&\quad-\frac{1}{2}\left(\alpha^{m+1}_j\right)^2\left|y^{m+1}_j\right|^2\tau^m_{j+1}-\alpha^{m+1}_j RM^{m+1}_j\tau^m_{j+1}-\left(\tau^m_{j+1},R^{m+1}_j\right)\tau^m_{j+1}.
\end{aligned}\tag{A.8}$$

In view of (3.21), (4.79) and (4.84) we may write

$$\begin{aligned}
&J^{m+1}_{j+1}\left(\tau^{m+1}_j,\tau^{m+1}_{j+1}\right)-J^{m+1}_j=J^{m+1}_{j+1}-J^{m+1}_j-\frac{1}{2}J^{m+1}_{j+1}\left|\tau^{m+1}_{j+1}-\tau^{m+1}_j\right|^2\\
&\quad=J^{m+1}_{j+1}-J^{m+1}_j-\frac{1}{2}\left(\alpha^{m+1}_j\right)^2\left|y^{m+1}_j\right|^2J^{m+1}_{j+1}-RS^{m+1}_jJ^{m+1}_{j+1}.
\end{aligned}\tag{A.9}$$

Taking into account (A.6)-(A.9) and using once more (4.84) for λ-term, the equation (A.5) translates into

$$\begin{aligned}
&\alpha^m_j\left(\tau^{m+1}_j,\frac{x^{m+1}_j-x^m_j}{\Delta t}\right)=\frac{1}{q^{m+1}_{j+1}}\left(y^{m+1}_{j+1}-y^{m+1}_j,\alpha^{m+1}_j y^{m+1}_j\right)\\
&\quad+\frac{1}{q^{m+1}_{j+1}}\left(y^{m+1}_{j+1}-y^{m+1}_j,R^{m+1}_j\right)\\
&\quad+\frac{1}{q^{m+1}_{j+1}}\left(y^{m+1}_{j+1}-y^{m+1}_j,\tau^m_{j+1}\right)\left(\tau^{m+1}_{j+1}-\tau^m_{j+1},\alpha^{m+1}_j y^{m+1}_j\right)\\
&\quad-\frac{1}{2}\frac{1}{q^{m+1}_{j+1}}\left(\alpha^{m+1}_j\right)^2\left|y^{m+1}_j\right|^2\left(y^{m+1}_{j+1}-y^{m+1}_j,\tau^m_{j+1}\right)\\
&\quad-\frac{1}{q^{m+1}_{j+1}}\left(\alpha^{m+1}_j RM^{m+1}_j+\left(\tau^m_{j+1},R^{m+1}_j\right)\right)\left(y^{m+1}_{j+1}-y^{m+1}_j,\tau^m_{j+1}\right)
\end{aligned}\tag{A.10}$$

$$
\begin{aligned}
&- \frac{1}{2} \frac{1}{q_{j+1}^{m+1}} \left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m+1} - \tau_{j+1}^{m} \right) \left| \tau_{j+1}^{m+1} - \tau_j^{m+1} \right|^2 \\
&- \frac{1}{4} \frac{1}{q_{j+1}^{m+1}} \left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m} \right) \left| \tau_{j+1}^{m+1} - \tau_j^{m+1} \right|^2 \left| \tau_{j+1}^{m+1} - \tau_{j+1}^{m} \right|^2 \\
&- \frac{1}{2} \left(J_{j+1}^{m+1} - J_j^{m+1} \right) + \frac{1}{4} \left(\alpha_j^{m+1} \right)^2 \left| y_j^{m+1} \right|^2 J_{j+1}^{m+1} + \frac{1}{2} RS_j^{m+1} J_{j+1}^{m+1} \\
&- \frac{\lambda}{2} \left(\alpha_j^{m+1} \right)^2 \left| y_j^{m+1} \right|^2 - \lambda \, RS_j^{m+1}.
\end{aligned}
$$

With the help of the elementary relation

$$
\left(y_{j+1}^{m+1}, y_j^{m+1} \right) = -\frac{1}{2} \left| y_{j+1}^{m+1} - y_j^{m+1} \right|^2 + \frac{1}{2} \left| y_{j+1}^{m+1} \right|^2 + \frac{1}{2} \left| y_j^{m+1} \right|^2
$$

we derive for the first term on the right-hand side of (A.10)

$$
\begin{aligned}
&\frac{1}{q_{j+1}^{m+1}} \left(y_{j+1}^{m+1} - y_j^{m+1}, \alpha_j^{m+1} y_j^{m+1} \right) = \frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}} \left(y_{j+1}^{m+1}, y_j^{m+1} \right) - \frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}} \left| y_j^{m+1} \right|^2 \\
&= \frac{1}{2} \frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}} \left(\left| y_{j+1}^{m+1} \right|^2 - \left| y_j^{m+1} \right|^2 \right) - \frac{1}{2} \frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}} \left| y_{j+1}^{m+1} - y_j^{m+1} \right|^2 .
\end{aligned}
\tag{A.11}
$$

For later use, we split the scalar product as

$$
\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m} \right) = \left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m+1} \right) + \left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m} - \tau_{j+1}^{m+1} \right) . \tag{A.12}
$$

Next, from (4.94) we derive

$$
\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m+1} \right) = -\frac{1}{2} \alpha_{j+1}^{m+1} \left| y_{j+1}^{m+1} \right|^2 - \frac{1}{2} \alpha_j^{m+1} \left| y_j^{m+1} \right|^2 - RP_{j+1}^{m+1} - RM_j^{m+1}. \tag{A.13}
$$

Combining (A.12)-(A.13) the fourth term on the right-hand side of (A.10) can be written in the following way

$$
\begin{aligned}
&-\frac{1}{2} \left(\alpha_j^{m+1} \right)^2 \left| y_j^{m+1} \right|^2 \frac{1}{q_{j+1}^{m+1}} \left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m} \right) \\
&= \frac{1}{4} \frac{\left(\alpha_j^{m+1} \right)^2}{q_{j+1}^{m+1}} \left| y_j^{m+1} \right|^2 \left(\alpha_{j+1}^{m+1} \left| y_{j+1}^{m+1} \right|^2 + \alpha_j^{m+1} \left| y_j^{m+1} \right|^2 \right) \\
&\quad + \frac{1}{2} \frac{\left(\alpha_j^{m+1} \right)^2}{q_{j+1}^{m+1}} \left| y_j^{m+1} \right|^2 \left(RP_{j+1}^{m+1} + RM_j^{m+1} \right) \\
&\quad - \frac{1}{2} \left(\alpha_j^{m+1} \right)^2 \left| y_j^{m+1} \right|^2 \frac{1}{q_{j+1}^{m+1}} \left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m} - \tau_{j+1}^{m+1} \right) ,
\end{aligned}
\tag{A.14}
$$

where RP_j^{m+1}, RM_j^{m+1} are given by (4.88).

In view of the calculations (A.11)-(A.14), equation (A.10) after dividing both sides by α_j^m takes the form

$$
\begin{aligned}
&\left(\tau_j^{m+1}, \frac{x_j^{m+1}-x_j^m}{\Delta t}\right) = \frac{1}{2}\frac{1}{\alpha_j^m}\frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}}\left(\left|y_{j+1}^{m+1}\right|^2 - \left|y_j^{m+1}\right|^2\right) \\
&- \frac{1}{2}\frac{1}{\alpha_j^m}\frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}}\left|y_{j+1}^{m+1}-y_j^{m+1}\right|^2 + \frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1}-y_j^{m+1}, R_j^{m+1}\right) \\
&+ \frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1}-y_j^{m+1}, \tau_{j+1}^m\right)\left(\tau_{j+1}^{m+1}-\tau_{j+1}^m, \alpha_j^{m+1}y_j^{m+1}\right) \\
&+ \frac{1}{4}\frac{\alpha_j^{m+1}}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_{j+1}^{m+1}\left|y_{j+1}^{m+1}\right|^2 + \alpha_j^{m+1}\left|y_j^{m+1}\right|^2\right) \\
&+ \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(RP_{j+1}^{m+1} + RM_j^{m+1}\right) \\
&- \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(y_{j+1}^{m+1}-y_j^{m+1}, \tau_{j+1}^m - \tau_{j+1}^{m+1}\right) \\
&- \frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(\alpha_j^{m+1}RM_j^{m+1} + \left(\tau_{j+1}^m, R_j^{m+1}\right)\right)\left(y_{j+1}^{m+1}-y_j^{m+1}, \tau_{j+1}^m\right) \\
&- \frac{1}{2}\frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1}-y_j^{m+1}, \tau_{j+1}^{m+1}-\tau_{j+1}^m\right)\left|\tau_{j+1}^{m+1}-\tau_j^{m+1}\right|^2 \\
&- \frac{1}{4}\frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1}-y_j^{m+1}, \tau_{j+1}^m\right)\left|\tau_{j+1}^{m+1}-\tau_j^{m+1}\right|^2\left|\tau_{j+1}^{m+1}-\tau_{j+1}^m\right|^2 \\
&- \frac{1}{2}\frac{1}{\alpha_j^m}\left(J_{j+1}^{m+1} - J_j^{m+1}\right) + \frac{1}{4}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2 J_{j+1}^{m+1} + \frac{1}{2}\frac{1}{\alpha_j^m}RS_j^{m+1}J_{j+1}^{m+1} \\
&- \frac{\lambda}{2}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2 - \lambda\frac{1}{\alpha_j^m}RS_j^{m+1}.
\end{aligned}
\tag{A.15}
$$

We note that only six terms from the right-hand side of (A.15) are used in the calculations in Lemma 4.11, while the remaining terms are estimated in Lemma 4.12. Therefore, it is convenient to write the equation (A.15) in a short form (A.1) with these chosen six terms. Thus, we include the first, second, fifth, eleventh, thirteenth and fourteenth terms into (A.15) in the full form. Furthermore, for brevity we denote by D_j^{m+1} all the terms that are left. The exact form of D_j^{m+1} is given in (A.2) in the formulation of the lemma.

□

In the following lemmas we perform similar calculations. Therefore, in order not to overburden the presentation, we omit several technical calculations concentrating only on the main results. The next lemma gives the representation for the second term on the right-hand side of (4.105).

Lemma A.2. *The second scalar product on the right-hand side of (4.105) takes the form*

$$
\begin{aligned}
\left(\tau_j^{m+1}, \frac{x_{j-1}^{m+1}-x_{j-1}^m}{\Delta t}\right) = &\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left(\left|y_{j-1}^{m+1}\right|^2-\left|y_{j-2}^{m+1}\right|^2\right)\\
&+\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}-y_{j-2}^{m+1}\right|^2\\
&-\frac{1}{4}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2+\alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right)\\
&-\frac{1}{2}\frac{1}{\alpha_{j-1}^m}\left(J_j^{m+1}-J_{j-1}^{m+1}\right)-\frac{1}{4}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 J_{j-1}^{m+1}\\
&+\frac{\lambda}{2}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2+E_j^{m+1},
\end{aligned}
\tag{A.16}
$$

where

$$
\begin{aligned}
E_j^{m+1} =&\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}, R_{j-1}^{m+1}\right)\\
&-\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(RP_{j-1}^{m+1}+RM_{j-2}^{m+1}\right)\\
&+\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}, \tau_{j-1}^m\right)\left(\tau_{j-1}^{m+1}-\tau_{j-1}^m, \alpha_{j-1}^{m+1}y_{j-1}^{m+1}\right)\\
&+\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}, \tau_{j-1}^m-\tau_{j-1}^{m+1}\right)\\
&+\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(\alpha_{j-1}^{m+1}RP_{j-1}^{m+1}-\left(\tau_{j-1}^m, R_{j-1}^{m+1}\right)\right)\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}, \tau_{j-1}^m\right)\\
&+\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}, \tau_{j-1}^{m+1}-\tau_{j-1}^m\right)\left|\tau_j^{m+1}-\tau_{j-1}^{m+1}\right|^2\\
&+\frac{1}{4}\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}, \tau_{j-1}^m\right)\left|\tau_{j-1}^{m+1}-\tau_{j-1}^m\right|^2\left|\tau_j^{m+1}-\tau_{j-1}^{m+1}\right|^2\\
&-\frac{1}{2}\frac{1}{\alpha_{j-1}^m}RS_{j-1}^{m+1}J_{j-1}^{m+1}+\frac{\lambda}{\alpha_{j-1}^m}RS_{j-1}^{m+1}
\end{aligned}
\tag{A.17}
$$

and R_j^{m+1}, RS_j^{m+1}, RM_j^{m+1}, RP_j^{m+1} *are given by (A.3).*

Proof. We take a scalar product between τ_j^{m+1} and equation (4.78) evaluated for $j-1$ and obtain

$$
\begin{aligned}
&\alpha_{j-1}^{m}\left(\tau_j^{m+1}, \frac{x_{j-1}^{m+1}-x_{j-1}^{m}}{\Delta t}\right)\\
&\quad=-\frac{\left(\tau_j^{m+1}, P_j^m\left(y_j^{m+1}-y_{j-1}^{m+1}\right)\right)}{q_j^{m+1}}+\frac{\left(\tau_j^{m+1}, P_{j-1}^m\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}\right)\right)}{q_{j-1}^{m+1}}\\
&\qquad-\frac{1}{4}\left(\left(\left|y_{j-1}^{m+1}\right|^2+\left|y_j^{m+1}\right|^2\right)\left(\tau_j^{m+1},\tau_j^{m+1}\right)-\left(\left|y_{j-2}^{m+1}\right|^2+\left|y_{j-1}^{m+1}\right|^2\right)\left(\tau_{j-1}^{m+1},\tau_j^{m+1}\right)\right)\\
&\qquad+\frac{\left(P_j^m\left(y_j^{m+1}-y_{j-1}^{m+1}\right),\tau_j^{m+1}\right)}{q_j^{m+1}}\left(\tau_j^{m+1},\tau_j^{m+1}\right)\\
&\qquad-\frac{\left(P_{j-1}^m\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}\right),\tau_{j-1}^{m+1}\right)}{q_{j-1}^{m+1}}\left(\tau_{j-1}^{m+1},\tau_j^{m+1}\right)+\lambda\left(\tau_j^{m+1}-\tau_{j-1}^{m+1},\tau_j^{m+1}\right).
\end{aligned}
$$

We observe that sum of the first and fourth terms on the right-hand side of the above equation is equal to zero. Next, using already introduced abbreviation (A.4) for J_j^{m+1} we arrive at

$$
\begin{aligned}
&\alpha_{j-1}^{m}\left(\tau_j^{m+1}, \frac{x_{j-1}^{m+1}-x_{j-1}^{m}}{\Delta t}\right)\\
&\quad=\frac{\left(P_{j-1}^m\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}\right),\tau_j^{m+1}\right)}{q_{j-1}^{m+1}}-\frac{1}{2}\left(J_j^{m+1}-J_{j-1}^{m+1}\left(\tau_{j-1}^{m+1},\tau_j^{m+1}\right)\right)\\
&\qquad-\frac{\left(P_{j-1}^m\left(y_{j-1}^{m+1}-y_{j-2}^{m+1}\right),\tau_{j-1}^{m+1}\right)}{q_{j-1}^{m+1}}\left(\tau_{j-1}^{m+1},\tau_j^{m+1}\right)+\lambda\left(\tau_j^{m+1}-\tau_{j-1}^{m+1},\tau_j^{m+1}\right).
\end{aligned}
\tag{A.18}
$$

Combining the first and third terms on the right-hand side of (A.18) and recalling (3.21) as well as the symmetry property of the projection matrix yield

$$
\begin{aligned}
&\frac{\left(P_{j-1}^m\tau_j^{m+1}, y_{j-1}^{m+1}-y_{j-2}^{m+1}\right)}{q_{j-1}^{m+1}}-\frac{\left(P_{j-1}^m\tau_{j-1}^{m+1}, y_{j-1}^{m+1}-y_{j-2}^{m+1}\right)}{q_{j-1}^{m+1}}\left(\tau_{j-1}^{m+1},\tau_j^{m+1}\right)\\
&\quad=\frac{\left(P_{j-1}^m\left(\tau_j^{m+1}-\tau_{j-1}^{m+1}\right), y_{j-1}^{m+1}-y_{j-2}^{m+1}\right)}{q_{j-1}^{m+1}}\\
&\qquad+\frac{1}{2}\frac{\left(P_{j-1}^m\tau_{j-1}^{m+1}, y_{j-1}^{m+1}-y_{j-2}^{m+1}\right)}{q_{j-1}^{m+1}}\left|\tau_j^{m+1}-\tau_{j-1}^{m+1}\right|^2.
\end{aligned}
\tag{A.19}
$$

Let us rewrite the above terms starting from the second one. Definition of the projection matrix (1.6) together with the identity (3.21) implies

$$
P_{j-1}^m\tau_{j-1}^{m+1}=\tau_{j-1}^{m+1}-\left(\tau_{j-1}^m,\tau_{j-1}^{m+1}\right)\tau_{j-1}^m=\tau_{j-1}^{m+1}-\tau_{j-1}^m+\frac{1}{2}\left|\tau_{j-1}^{m+1}-\tau_{j-1}^m\right|^2\tau_{j-1}^m. \tag{A.20}
$$

Following the ideas in (A.8) we reformulate the next term as

$$
\begin{aligned}
P_{j-1}^{m}\left(\tau_{j}^{m+1}-\tau_{j-1}^{m+1}\right)=&\alpha_{j-1}^{m+1}y_{j-1}^{m+1}+R_{j-1}^{m+1}+\left(\tau_{j-1}^{m+1}-\tau_{j-1}^{m},\alpha_{j-1}^{m+1}y_{j-1}^{m+1}\right)\tau_{j-1}^{m}\\
&+\frac{1}{2}\left(\alpha_{j-1}^{m+1}\right)^{2}\left|y_{j-1}^{m+1}\right|^{2}\tau_{j-1}^{m}+\alpha_{j-1}^{m+1}RP_{j-1}^{m+1}\tau_{j-1}^{m}\\
&-\left(\tau_{j-1}^{m},R_{j-1}^{m+1}\right)\tau_{j-1}^{m}.
\end{aligned}
\tag{A.21}
$$

Analogously to (A.9), we obtain

$$
\begin{aligned}
J_{j}^{m+1}-J_{j-1}^{m+1}\left(\tau_{j-1}^{m+1},\tau_{j}^{m+1}\right)=&J_{j}^{m+1}-J_{j-1}^{m+1}+\frac{1}{2}\left(\alpha_{j-1}^{m+1}\right)^{2}\left|y_{j-1}^{m+1}\right|^{2}J_{j-1}^{m+1}\\
&+RS_{j-1}^{m+1}J_{j-1}^{m+1}
\end{aligned}
\tag{A.22}
$$

In view of (A.19)-(A.22) and applying (4.84) for λ-term, the equation (A.18) takes the form

$$
\begin{aligned}
\alpha_{j-1}^{m}&\left(\tau_{j}^{m+1},\frac{x_{j-1}^{m+1}-x_{j-1}^{m}}{\Delta t}\right)=\frac{1}{q_{j-1}^{m+1}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\alpha_{j-1}^{m+1}y_{j-1}^{m+1}\right)\\
&+\frac{1}{q_{j-1}^{m+1}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},R_{j-1}^{m+1}\right)\\
&+\frac{1}{q_{j-1}^{m+1}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\left(\tau_{j-1}^{m+1}-\tau_{j-1}^{m},\alpha_{j-1}^{m+1}y_{j-1}^{m+1}\right)\\
&+\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\left(\alpha_{j-1}^{m+1}\right)^{2}\left|y_{j-1}^{m+1}\right|^{2}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\\
&+\frac{1}{q_{j-1}^{m+1}}\left(\alpha_{j-1}^{m+1}RP_{j-1}^{m+1}-\left(\tau_{j-1}^{m},R_{j-1}^{m+1}\right)\right)\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\\
&+\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m+1}-\tau_{j-1}^{m}\right)\left|\tau_{j}^{m+1}-\tau_{j-1}^{m+1}\right|^{2}\\
&+\frac{1}{4}\frac{1}{q_{j-1}^{m+1}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\left|\tau_{j}^{m+1}-\tau_{j-1}^{m+1}\right|^{2}\left|\tau_{j-1}^{m+1}-\tau_{j-1}^{m}\right|^{2}\\
&-\frac{1}{2}\left(J_{j}^{m+1}-J_{j-1}^{m+1}\right)-\frac{1}{4}\left(\alpha_{j-1}^{m+1}\right)^{2}\left|y_{j-1}^{m+1}\right|^{2}J_{j-1}^{m+1}-\frac{1}{2}RS_{j-1}^{m+1}J_{j-1}^{m+1}\\
&+\frac{\lambda}{2}\left(\alpha_{j-1}^{m+1}\right)^{2}\left|y_{j-1}^{m+1}\right|^{2}+\lambda RS_{j-1}^{m+1}.
\end{aligned}
\tag{A.23}
$$

In a similar way as it is done in Lemma A.1 we get for the first term on the right-hand side of (A.23)

$$
\frac{1}{q_{j-1}^{m+1}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\alpha_{j-1}^{m+1}y_{j-1}^{m+1}\right)=\frac{1}{2}\frac{\alpha_{j-1}^{m+1}}{q_{j-1}^{m+1}}\left(\left|y_{j-1}^{m+1}\right|^{2}-\left|y_{j-2}^{m+1}\right|^{2}\right)+\frac{1}{2}\frac{\alpha_{j-1}^{m+1}}{q_{j-1}^{m+1}}\left|y_{j-1}^{m+1}-y_{j-2}^{m+1}\right|^{2}.
$$

Using (4.94) and (4.88) we rewrite the fourth term on the right-hand side of (A.23) as

$$
\begin{aligned}
&\frac{1}{2}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{q_{j-1}^{m+1}}\left|y_{j-1}^{m+1}\right|^2\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\\
&=-\frac{1}{4}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{q_{j-1}^{m+1}}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2+\alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right)-\frac{1}{2}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{q_{j-1}^{m+1}}\left|y_{j-1}^{m+1}\right|^2 RP_{j-1}^{m+1}\\
&\quad-\frac{1}{2}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{q_{j-1}^{m+1}}\left|y_{j-1}^{m+1}\right|^2 RM_{j-2}^{m+1}+\frac{1}{2}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{q_{j-1}^{m+1}}\left|y_{j-1}^{m+1}\right|^2\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}-\tau_{j-1}^{m+1}\right).
\end{aligned}
$$

Taking into account the above calculations the equation (A.23) after dividing both sides by α_{j-1}^{m} translates into

$$
\begin{aligned}
&\left(\tau_j^{m+1},\frac{x_{j-1}^{m+1}-x_{j-1}^{m}}{\Delta t}\right)=\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^{m}}\left(\left|y_{j-1}^{m+1}\right|^2-\left|y_{j-2}^{m+1}\right|^2\right)\\
&+\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^{m}}\left|y_{j-1}^{m+1}-y_{j-2}^{m+1}\right|^2+\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^{m}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},R_{j-1}^{m+1}\right)\\
&+\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^{m}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\left(\tau_{j-1}^{m+1}-\tau_{j-1}^{m},\alpha_{j-1}^{m+1}y_{j-1}^{m+1}\right)\\
&-\frac{1}{4}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^{m}}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2+\alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right)\\
&-\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^{m}}\left|y_{j-1}^{m+1}\right|^2\left(RP_{j-1}^{m+1}+RM_{j-2}^{m+1}\right)\\
&+\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^{m}}\left|y_{j-1}^{m+1}\right|^2\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}-\tau_{j-1}^{m+1}\right)\\
&+\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^{m}}\left(\alpha_{j-1}^{m+1}RP_{j-1}^{m+1}-\left(\tau_{j-1}^{m},R_{j-1}^{m+1}\right)\right)\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\\
&+\frac{1}{2}\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^{m}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m+1}-\tau_{j-1}^{m}\right)\left|\tau_j^{m+1}-\tau_{j-1}^{m+1}\right|^2\\
&+\frac{1}{4}\frac{1}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^{m}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\left|\tau_j^{m+1}-\tau_{j-1}^{m+1}\right|^2\left|\tau_{j-1}^{m+1}-\tau_{j-1}^{m}\right|^2\\
&-\frac{1}{2}\frac{1}{\alpha_{j-1}^{m}}\left(J_j^{m+1}-J_{j-1}^{m+1}\right)-\frac{1}{4}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^{m}}\left|y_{j-1}^{m+1}\right|^2 J_{j-1}^{m+1}-\frac{1}{2}\frac{1}{\alpha_{j-1}^{m}}RS_{j-1}^{m+1}J_{j-1}^{m+1}\\
&+\frac{\lambda}{2}\frac{\left(\alpha_{j-1}^{m+1}\right)^2}{\alpha_{j-1}^{m}}\left|y_{j-1}^{m+1}\right|^2+\frac{\lambda}{\alpha_{j-1}^{m}}RS_{j-1}^{m+1}.
\end{aligned}
\tag{A.24}
$$

Again, we write the equation (A.24) in a short form (A.16) denoting by E_j^{m+1} the remaining terms given in (A.17).

□

Lemma A.3. *The third term on the right-hand side of (4.105) has the form*

$$
\begin{aligned}
&\frac{1}{2}\left(y_j^{m+1}, \frac{x_j^{m+1}-x_j^m}{\Delta t}\right) q_j^{m+1} \\
&= -\frac{1}{4}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(\left|y_{j+1}^{m+1}\right|^2 - \left|y_j^{m+1}\right|^2\right) + \frac{1}{4}\frac{1}{\alpha_j^m}\left(\left|y_j^{m+1}\right|^2 - \left|y_{j-1}^{m+1}\right|^2\right) \\
&\quad + \frac{1}{4}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left|y_{j+1}^{m+1} - y_j^{m+1}\right|^2 + \frac{1}{4}\frac{1}{\alpha_j^m}\left|y_j^{m+1} - y_{j-1}^{m+1}\right|^2 \\
&\quad - \frac{1}{8}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 + \alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2\right) \\
&\quad - \frac{1}{8}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\alpha_{j+1}^{m+1}\left|y_{j+1}^{m+1}\right|^2 + \alpha_j^{m+1}\left|y_j^{m+1}\right|^2\right) \\
&\quad - \frac{1}{8}q_j^{m+1}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(J_{j+1}^{m+1} + J_j^{m+1}\right) + \frac{\lambda}{2}q_j^{m+1}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 + F_j^{m+1},
\end{aligned}
\tag{A.25}
$$

where

$$
\begin{aligned}
F_j^{m+1} =& -\frac{1}{4}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 RP_{j+1}^{m+1} - \frac{1}{4}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 RM_j^{m+1} \\
&+ \frac{1}{4}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_{j+1}^m - \tau_{j+1}^{m+1}, y_{j+1}^{m+1} - y_j^{m+1}\right) \\
&+ \frac{1}{2}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}RM_j^{m+1}\left(\tau_{j+1}^m, y_{j+1}^{m+1} - y_j^{m+1}\right) \\
&+ \frac{1}{2}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(\tau_{j+1}^m - \tau_{j+1}^{m+1}, y_j^{m+1}\right)\left(\tau_{j+1}^m, y_{j+1}^{m+1} - y_j^{m+1}\right) \\
&- \frac{1}{4}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 RP_j^{m+1} - \frac{1}{4}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 RM_{j-1}^{m+1} \\
&+ \frac{1}{4}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_j^m - \tau_j^{m+1}, y_j^{m+1} - y_{j-1}^{m+1}\right) \\
&+ \frac{1}{2}\frac{1}{\alpha_j^m}RP_j^{m+1}\left(\tau_j^m, y_j^{m+1} - y_{j-1}^{m+1}\right) \\
&- \frac{1}{2}\frac{1}{\alpha_j^m}\left(\tau_j^m - \tau_j^{m+1}, y_j^{m+1}\right)\left(\tau_j^m, y_j^{m+1} - y_{j-1}^{m+1}\right) \\
&- \frac{1}{4}\frac{q_j^{m+1}}{\alpha_j^m}RM_j^{m+1}J_{j+1}^{m+1} - \frac{1}{4}\frac{q_j^{m+1}}{\alpha_j^m}RP_j^{m+1}J_j^{m+1} \\
&+ \frac{1}{2}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m+1} - \tau_{j+1}^m\right)\left(\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 + RM_j^{m+1}\right)
\end{aligned}
\tag{A.26}
$$

$$
\begin{aligned}
&+\frac{1}{4}\frac{q_j^{m+1}}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(y_{j+1}^{m+1}-y_j^{m+1},\tau_{j+1}^m\right)\left|\tau_{j+1}^{m+1}-\tau_{j+1}^m\right|^2\left(\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2+RM_j^{m+1}\right)\\
&-\frac{1}{2}\frac{1}{\alpha_j^m}\left(y_j^{m+1}-y_{j-1}^{m+1},\tau_j^{m+1}-\tau_j^m\right)\left(-\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2-RP_j^{m+1}\right)\\
&-\frac{1}{4}\frac{1}{\alpha_j^m}\left(y_j^{m+1}-y_{j-1}^{m+1},\tau_j^m\right)\left|\tau_j^{m+1}-\tau_j^m\right|^2\left(-\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2-RP_j^{m+1}\right)\\
&+\frac{\lambda}{2}\frac{q_j^{m+1}}{\alpha_j^m}RM_j^{m+1}+\frac{\lambda}{2}\frac{q_j^{m+1}}{\alpha_j^m}RP_j^{m+1}
\end{aligned}
$$

with RP_j^{m+1} *and* RM_j^{m+1} *given by (A.3).*

Proof. Equation (4.78) after taking a scalar product with y_j^{m+1} and recalling the symmetry of the projection matrix turns into

$$
\begin{aligned}
&\alpha_j^m\left(y_j^{m+1},\frac{x_j^{m+1}-x_j^m}{\Delta t}\right)\\
&=-\frac{1}{q_{j+1}^{m+1}}\left(P_{j+1}^m y_j^{m+1},y_{j+1}^{m+1}-y_j^{m+1}\right)+\frac{1}{q_j^{m+1}}\left(P_j^m y_j^{m+1},y_j^{m+1}-y_{j-1}^{m+1}\right)\\
&\quad-\frac{1}{4}\Big(\left(\left|y_j^{m+1}\right|^2+\left|y_{j+1}^{m+1}\right|^2\right)\left(y_j^{m+1},\tau_{j+1}^{m+1}\right)\\
&\qquad\quad-\left(\left|y_{j-1}^{m+1}\right|^2+\left|y_j^{m+1}\right|^2\right)\left(y_j^{m+1},\tau_j^{m+1}\right)\Big)\\
&\quad+\frac{\left(P_{j+1}^m\tau_{j+1}^{m+1},y_{j+1}^{m+1}-y_j^{m+1}\right)}{q_{j+1}^{m+1}}\left(\tau_{j+1}^{m+1},y_j^{m+1}\right)\\
&\quad-\frac{\left(P_j^m\tau_j^{m+1},y_j^{m+1}-y_{j-1}^{m+1}\right)}{q_j^{m+1}}\left(\tau_j^{m+1},y_j^{m+1}\right)+\lambda\left(\tau_{j+1}^{m+1}-\tau_j^{m+1},y_j^{m+1}\right).
\end{aligned}
\tag{A.27}
$$

First, with the help of (4.86) we obtain

$$
\begin{aligned}
P_j^m y_j^{m+1}=&y_j^{m+1}-\left(\tau_j^m,y_j^{m+1}\right)\tau_j^m=y_j^{m+1}-\left(\tau_j^{m+1},y_j^{m+1}\right)\tau_j^m-\left(\tau_j^m-\tau_j^{m+1},y_j^{m+1}\right)\tau_j^m\\
=&y_j^{m+1}+\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2\tau_j^m+RP_j^{m+1}\tau_j^m-\left(\tau_j^m-\tau_j^{m+1},y_j^{m+1}\right)\tau_j^m.
\end{aligned}
$$

Whereas the identity (4.85) implies

$$
\begin{aligned}
P_{j+1}^m y_j^{m+1}=&y_j^{m+1}-\left(\tau_{j+1}^m,y_j^{m+1}\right)\tau_{j+1}^m\\
=&y_j^{m+1}-\left(\tau_{j+1}^{m+1},y_j^{m+1}\right)\tau_{j+1}^m-\left(\tau_{j+1}^m-\tau_{j+1}^{m+1},y_j^{m+1}\right)\tau_{j+1}^m\\
=&y_j^{m+1}-\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2\tau_{j+1}^m-RM_j^{m+1}\tau_{j+1}^m-\left(\tau_{j+1}^m-\tau_{j+1}^{m+1},y_j^{m+1}\right)\tau_{j+1}^m.
\end{aligned}
$$

Analogously to (A.7), we derive

$$
P_j^m\tau_j^{m+1}=\tau_j^{m+1}-\left(\tau_j^m,\tau_j^{m+1}\right)\tau_j^m=\tau_j^{m+1}-\tau_j^m+\frac{1}{2}\left|\tau_j^{m+1}-\tau_j^m\right|^2\tau_j^m.
$$

Using the relations (4.85), (4.86), abbreviation (A.4) and calculated above products between projection matrix on the one side and tangent and curvature vectors on the other side, the equation (A.27) after dividing both sides by α_j^m takes the form

$$
\begin{aligned}
&\left(y_j^{m+1}, \frac{x_j^{m+1}-x_j^m}{\Delta t}\right) = -\frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(y_j^{m+1}, y_{j+1}^{m+1}-y_j^{m+1}\right) \\
&+\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_{j+1}^m, y_{j+1}^{m+1}-y_j^{m+1}\right) \\
&+\frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}RM_j^{m+1}\left(\tau_{j+1}^m, y_{j+1}^{m+1}-y_j^{m+1}\right) \\
&+\frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(\tau_{j+1}^m-\tau_{j+1}^{m+1}, y_j^{m+1}\right)\left(\tau_{j+1}^m, y_{j+1}^{m+1}-y_j^{m+1}\right) \\
&+\frac{1}{q_j^{m+1}}\frac{1}{\alpha_j^m}\left(y_j^{m+1}, y_j^{m+1}-y_{j-1}^{m+1}\right)+\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_j^m, y_j^{m+1}-y_{j-1}^{m+1}\right) \\
&+\frac{1}{q_j^{m+1}}\frac{1}{\alpha_j^m}RP_j^{m+1}\left(\tau_j^m, y_j^{m+1}-y_{j-1}^{m+1}\right) \\
&-\frac{1}{q_j^{m+1}}\frac{1}{\alpha_j^m}\left(\tau_j^m-\tau_j^{m+1}, y_j^{m+1}\right)\left(\tau_j^m, y_j^{m+1}-y_{j-1}^{m+1}\right)-\frac{1}{4}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 J_{j+1}^{m+1} \\
&-\frac{1}{2}\frac{1}{\alpha_j^m}RM_j^{m+1}J_{j+1}^{m+1}-\frac{1}{4}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 J_j^{m+1}-\frac{1}{2}\frac{1}{\alpha_j^m}RP_j^{m+1}J_j^{m+1} \\
&+\frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(y_{j+1}^{m+1}-y_j^{m+1}, \tau_{j+1}^{m+1}-\tau_{j+1}^m\right)\left(\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2+RM_j^{m+1}\right) \\
&+\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(y_{j+1}^{m+1}-y_j^{m+1}, \tau_{j+1}^m\right)\left|\tau_{j+1}^{m+1}-\tau_{j+1}^m\right|^2\left(\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2+RM_j^{m+1}\right) \\
&-\frac{1}{q_j^{m+1}}\frac{1}{\alpha_j^m}\left(y_j^{m+1}-y_{j-1}^{m+1}, \tau_j^{m+1}-\tau_j^m\right)\left(-\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2-RP_j^{m+1}\right) \\
&-\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{1}{\alpha_j^m}\left(y_j^{m+1}-y_{j-1}^{m+1}, \tau_j^m\right)\left|\tau_j^{m+1}-\tau_j^m\right|^2\left(-\frac{1}{2}\alpha_j^{m+1}\left|y_j^{m+1}\right|^2-RP_j^{m+1}\right) \\
&+\lambda\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2+\lambda\frac{1}{\alpha_j^m}RM_j^{m+1}+\lambda\frac{1}{\alpha_j^m}RP_j^{m+1}.
\end{aligned}
\tag{A.28}
$$

The first scalar product on the right-hand side of (A.28) after completing a square becomes

$$
\begin{aligned}
-\frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(y_j^{m+1}, y_{j+1}^{m+1}-y_j^{m+1}\right) = &-\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(\left|y_{j+1}^{m+1}\right|^2-\left|y_j^{m+1}\right|^2\right) \\
&+\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left|y_{j+1}^{m+1}-y_j^{m+1}\right|^2,
\end{aligned}
$$

while the second scalar product in view of (4.85) and (4.86) can be transformed as follows

$$\begin{aligned}
&\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_{j+1}^m, y_{j+1}^{m+1}-y_j^{m+1}\right) = \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_{j+1}^{m+1}, y_{j+1}^{m+1}-y_j^{m+1}\right)\\
&\quad+\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_{j+1}^m-\tau_{j+1}^{m+1}, y_{j+1}^{m+1}-y_j^{m+1}\right)\\
&=-\frac{1}{4}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\alpha_{j+1}^{m+1}\left|y_j^{m+1}\right|^2\left|y_{j+1}^{m+1}\right|^2-\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 RP_{j+1}^{m+1}\\
&\quad-\frac{1}{4}\frac{1}{q_{j+1}^{m+1}}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^4-\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 RM_j^{m+1}\\
&\quad+\frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_{j+1}^m-\tau_{j+1}^{m+1}, y_{j+1}^{m+1}-y_j^{m+1}\right).
\end{aligned}$$

We note that the first and the third terms on the right-hand side of the above equality will be used later in a short representation of the equation (A.28).
For the fifth term on the right-hand side of (A.28) we obtain

$$\begin{aligned}
\frac{1}{q_j^{m+1}}\frac{1}{\alpha_j^m}\left(y_j^{m+1}, y_j^{m+1}-y_{j-1}^{m+1}\right) =&\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{1}{\alpha_j^m}\left(\left|y_j^{m+1}\right|^2-\left|y_{j-1}^{m+1}\right|^2\right)\\
&+\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{1}{\alpha_j^m}\left|y_j^{m+1}-y_{j-1}^{m+1}\right|^2.
\end{aligned}$$

And the sixth scalar product gives

$$\begin{aligned}
&\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_j^m, y_j^{m+1}-y_{j-1}^{m+1}\right) = \frac{1}{2}\frac{1}{q_j^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_j^{m+1}, y_j^{m+1}-y_{j-1}^{m+1}\right)\\
&\quad+\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_j^m-\tau_j^{m+1}, y_j^{m+1}-y_{j-1}^{m+1}\right)\\
&=-\frac{1}{4}\frac{1}{q_j^{m+1}}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^4-\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 RP_j^{m+1}\\
&\quad-\frac{1}{4}\frac{1}{q_j^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2\left|y_j^{m+1}\right|^2-\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2 RM_{j-1}^{m+1}\\
&\quad+\frac{1}{2}\frac{1}{q_j^{m+1}}\frac{\alpha_j^{m+1}}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(\tau_j^m-\tau_j^{m+1}, y_j^{m+1}-y_{j-1}^{m+1}\right),
\end{aligned}$$

from here again the first and the third terms are of our interest.
Collecting the above results the equation (A.28) after multiplying it by $\frac{1}{2}q_j^{m+1}$ in a short form will look like (A.25). In F_j^{m+1} we include all the terms that are left after this concise formulation.

□

Lemma A.4. *The fourth scalar product on the right-hand side of (4.105) transforms into*

$$\begin{aligned}
&\frac{1}{2}\left(y_{j-1}^{m+1}, \frac{x_{j-1}^{m+1}-x_{j-1}^{m}}{\Delta t}\right) q_j^{m+1} \\
&= -\frac{1}{4}\frac{1}{\alpha_{j-1}^m}\left(\left|y_j^{m+1}\right|^2 - \left|y_{j-1}^{m+1}\right|^2\right) + \frac{1}{4}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(\left|y_{j-1}^{m+1}\right|^2 - \left|y_{j-2}^{m+1}\right|^2\right) \\
&\quad + \frac{1}{4}\frac{1}{\alpha_{j-1}^m}\left|y_j^{m+1} - y_{j-1}^{m+1}\right|^2 + \frac{1}{4}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1} - y_{j-2}^{m+1}\right|^2 \\
&\quad - \frac{1}{8}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2 + \alpha_{j-2}^{m+1}\left|y_{j-2}^{m+1}\right|^2\right) \\
&\quad - \frac{1}{8}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\alpha_j^{m+1}\left|y_j^{m+1}\right|^2 + \alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2\right) \\
&\quad - \frac{1}{8}q_j^{m+1}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(J_j^{m+1} + J_{j-1}^{m+1}\right) + \frac{\lambda}{2}q_j^{m+1}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 + G_j^{m+1},
\end{aligned} \tag{A.29}$$

where

$$\begin{aligned}
G_j^{m+1} = &-\frac{1}{4}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 RP_j^{m+1} - \frac{1}{4}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 RM_{j-1}^{m+1} \\
&+ \frac{1}{4}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\tau_j^m - \tau_j^{m+1}, y_j^{m+1} - y_{j-1}^{m+1}\right) \\
&+ \frac{1}{2}\frac{1}{\alpha_{j-1}^m}RM_{j-1}^{m+1}\left(\tau_j^m, y_j^{m+1} - y_{j-1}^{m+1}\right) \\
&+ \frac{1}{2}\frac{1}{\alpha_{j-1}^m}\left(\tau_j^m - \tau_j^{m+1}, y_{j-1}^{m+1}\right)\left(\tau_j^m, y_j^{m+1} - y_{j-1}^{m+1}\right) \\
&+ \frac{1}{2}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(y_{j-1}^{m+1}, y_{j-1}^{m+1} - y_{j-2}^{m+1}\right) - \frac{1}{4}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 RP_{j-1}^{m+1} \\
&- \frac{1}{4}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2 RM_{j-2}^{m+1} \\
&+ \frac{1}{4}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{\alpha_{j-1}^{m+1}}{\alpha_{j-1}^m}\left|y_{j-1}^{m+1}\right|^2\left(\tau_{j-1}^m - \tau_{j-1}^{m+1}, y_{j-1}^{m+1} - y_{j-2}^{m+1}\right) \\
&+ \frac{1}{2}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}RP_{j-1}^{m+1}\left(\tau_{j-1}^m, y_{j-1}^{m+1} - y_{j-2}^{m+1}\right) \\
&- \frac{1}{2}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^m}\left(\tau_{j-1}^m - \tau_{j-1}^{m+1}, y_{j-1}^{m+1}\right)\left(\tau_{j-1}^m, y_{j-1}^{m+1} - y_{j-2}^{m+1}\right) \\
&- \frac{1}{4}\frac{q_j^{m+1}}{\alpha_{j-1}^m}RM_{j-1}^{m+1}J_j^{m+1} - \frac{1}{4}\frac{q_j^{m+1}}{\alpha_{j-1}^m}RP_{j-1}^{m+1}J_{j-1}^{m+1}
\end{aligned} \tag{A.30}$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{1}{\alpha_{j-1}^{m}}\left(y_j^{m+1}-y_{j-1}^{m+1},\tau_j^{m+1}-\tau_j^{m}\right)\left(\frac{1}{2}\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2+RM_{j-1}^{m+1}\right)\\
&+\frac{1}{4}\frac{1}{\alpha_{j-1}^{m}}\left(y_j^{m+1}-y_{j-1}^{m+1},\tau_j^{m}\right)\left|\tau_j^{m+1}-\tau_j^{m}\right|^2\left(\frac{1}{2}\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2+RM_{j-1}^{m+1}\right)\\
&-\frac{1}{2}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^{m}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m+1}-\tau_{j-1}^{m}\right)\left(-\frac{1}{2}\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2-RP_{j-1}^{m+1}\right)\\
&-\frac{1}{4}\frac{q_j^{m+1}}{q_{j-1}^{m+1}}\frac{1}{\alpha_{j-1}^{m}}\left(y_{j-1}^{m+1}-y_{j-2}^{m+1},\tau_{j-1}^{m}\right)\left|\tau_{j-1}^{m+1}-\tau_{j-1}^{m}\right|^2\left(-\frac{1}{2}\alpha_{j-1}^{m+1}\left|y_{j-1}^{m+1}\right|^2-RP_{j-1}^{m+1}\right)\\
&+\frac{\lambda}{2}\frac{q_j^{m+1}}{\alpha_{j-1}^{m}}RM_{j-1}^{m+1}+\frac{\lambda}{2}\frac{q_j^{m+1}}{\alpha_{j-1}^{m}}RP_{j-1}^{m+1}
\end{aligned}
$$

with RP_j^{m+1} and RM_j^{m+1} given by (A.3).

Proof. Replacing j with $j-1$ in the derivation of Lemma A.3 and multiplying the result by $\frac{1}{2}q_j^{m+1}$ at the end of the proof we obtain the claim of the lemma. $\square$

Appendix B

We derive here the estimate of the remainder term required for the proof in Lemma 4.12.

Lemma B.1 (Estimation of Rem_j^{m+1})**.** *The remainder term* Rem_j^{m+1}*,* $m = 0, ..., M-1$*,* $j = 1, ..., N$ *can be estimated in the following way*

$$\begin{aligned} \left|Rem_j^{m+1}\right| \leq & Ch^{-2}\left(1 + \left\|y_{hu}^{m+1}\right\|_{L^\infty}\right) \sum_{k=0}^{m} \left\|\tau_h^{k+1} - \tau_h^k\right\|_{L^2(I_{(j)})}^2 \\ & + Ch^{-\frac{1}{2}} \left\|y_{hu}^{m+1}\right\|_{L^\infty} \left\|\tau_h^{m+1} - \tau_h^m\right\|_{L^2(I_{(j)})}, \end{aligned} \tag{B.1}$$

where $I_{(j)}$ *is given by (4.113).*

Proof. To begin, we recall the definition (4.103) of Rem_j^{m+1}

$$Rem_j^{m+1} = D_j^{m+1} - E_j^{m+1} + F_j^{m+1} + G_j^{m+1},$$

where D_j^{m+1}, E_j^{m+1}, F_j^{m+1}, G_j^{m+1} are given by (A.2), (A.17), (A.26) and (A.30), respectively. Let us analyze these terms. From the proofs of Lemma A.1-Lemma A.4 follows that that the first two and the last two terms have a similar structure. The only difference within a pair, that one observes, is a subinterval I_j, on which the discrete expressions are taken, where index j varies from $j-2$ to $j+2$. Therefore, we may concentrate only on one term from each group using the notation $I_{(j)}$ for the resulting estimate of Rem_j^{m+1}. Running a few steps forward both groups produce the same estimate. Hence, we choose D_j^{m+1} from the first pair and repeat here for convenience its representation

$$\begin{aligned} D_j^{m+1} = & \frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1} - y_j^{m+1}, R_j^{m+1}\right) + \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(RP_{j+1}^{m+1} + RM_j^{m+1}\right) \\ & + \frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^m\right)\left(\tau_{j+1}^{m+1} - \tau_{j+1}^m, \alpha_j^{m+1} y_j^{m+1}\right) \\ & - \frac{1}{2}\frac{1}{q_{j+1}^{m+1}}\frac{\left(\alpha_j^{m+1}\right)^2}{\alpha_j^m}\left|y_j^{m+1}\right|^2\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^m - \tau_{j+1}^{m+1}\right) \\ & - \frac{1}{q_{j+1}^{m+1}}\frac{1}{\alpha_j^m}\left(\alpha_j^{m+1} RM_j^{m+1} + \left(\tau_{j+1}^m, R_j^{m+1}\right)\right)\left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^m\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1}-y_j^{m+1},\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right)\left|\tau_{j+1}^{m+1}-\tau_j^{m+1}\right|^2\\
&-\frac{1}{4}\frac{1}{\alpha_j^m}\frac{1}{q_{j+1}^{m+1}}\left(y_{j+1}^{m+1}-y_j^{m+1},\tau_{j+1}^{m}\right)\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|^2\left|\tau_{j+1}^{m+1}-\tau_j^{m+1}\right|^2\\
&+\frac{1}{4}\frac{1}{\alpha_j^m}RS_j^{m+1}\left(\left|y_j^{m+1}\right|^2+\left|y_{j+1}^{m+1}\right|^2\right)-\lambda\frac{1}{\alpha_j^m}RS_j^{m+1}\\
=&\sum_{i=1}^{9}Q_i,
\end{aligned}
$$

where RP_j^{m+1}, RM_j^{m+1}, RS_j^{m+1}, R_j^{m+1} presented in (A.3). In order to simplify the estimation of D_j^{m+1}, we keep the above abbreviations. Moreover, it will be helpful to express them in terms of R_j^{m+1}. To begin, we recall the estimate (4.97)

$$
\left|RP_j^{m+1}\right|,\left|RM_j^{m+1}\right|\le Ch^{-1}\left|R_j^{m+1}\right|.
$$

Thus, it remains to consider RS_j^{m+1}. Using the definitions (4.87), (4.88) of RS_j^{m+1} and RM_j^{m+1}, respectively, we can express RS_j^{m+1} as

$$
RS_j^{m+1}=\alpha_j^{m+1}RM_j^{m+1}+\left(\tau_{j+1}^{m+1},R_j^{m+1}\right).
$$

Next, in view of the definition (4.80) of α_j^m and (4.16) we deduce

$$
\frac{1}{4}c_0h_j\le q_j^m\le 4C_0h_j,\quad m=0,...,M,\quad j=1,...,N,
$$

from what the estimate follows

$$
\left|RS_j^{m+1}\right|\le C\left|R_j^{m+1}\right|.
$$

With the help of (4.84) we can rewrite the difference of the tangents as

$$
\left|\tau_{j+1}^{m+1}-\tau_j^{m+1}\right|^2=\left(\alpha_j^{m+1}\right)^2\left|y_j^{m+1}\right|^2+2RS_j^{m+1},
$$

what yields the splitting of the terms in the way $Q_6=Q_{6,1}+Q_{6,2}$, $Q_7=Q_{7,1}+Q_{7,2}$. Let us next estimate D_j^{m+1} combining the terms with a similar structure

$$
\begin{aligned}
\left|Q_1\right|,\left|Q_5\right|\le&Ch^{-1}\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|R_j^{m+1}\right|,\\
\left|Q_2\right|\le&Ch^{-1}\left(\left|R_j^{m+1}\right|+\left|R_{j+1}^{m+1}\right|\right),\\
\left|Q_3\right|\le&C\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|,\\
\left|Q_4\right|,\left|Q_{6,1}\right|\le&Ch\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|,\\
\left|Q_{6,2}\right|\le&Ch^{-1}\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|\left|R_j^{m+1}\right|,\\
\left|Q_{7,1}\right|\le&Ch\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|^2,\\
\left|Q_{7,2}\right|\le&Ch^{-1}\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|^2\left|R_j^{m+1}\right|,\\
\left|Q_8\right|,\left|Q_9\right|\le&Ch^{-1}\left|R_j^{m+1}\right|.
\end{aligned}
$$

For ease of presentation, we restrict ourselves to the terms of lower order only. If one allows a crude estimate for $Q_{6,2}$ and $Q_{7,2}$ estimating the difference of the tangents by a constant one obtains then Q_1.For the same reason, we may ignore $Q_4, Q_{6,1}$ as well as $Q_{7,1}$ and consider Q_3 instead. Furthermore, we choose Q_2 and neglect Q_8 and Q_9. Hence, we have to deal with three terms Q_1, Q_2, Q_3. Using an inverse inequality (2.12) we find

$$|Q_3| \le Ch^{-\frac{1}{2}} \left\| y_{hu}^{m+1} \right\|_{L^\infty} \left\| \tau_h^{m+1} - \tau_h^m \right\|_{L^2(I_{j+1})}.$$

Thus, combining the estimates for Q_1, Q_2 and Q_3 we deduce

$$\begin{aligned} \left| D_j^{m+1} \right| \le & Ch^{-1} \left(1 + \left\| y_{hu}^{m+1} \right\|_{L^\infty} \right) \left(\left| R_j^{m+1} \right| + \left| R_{j+1}^{m+1} \right| \right) \\ & + Ch^{-\frac{1}{2}} \left\| y_{hu}^{m+1} \right\|_{L^\infty} \left\| \tau_h^{m+1} - \tau_h^m \right\|_{L^2(I_{(j)})}. \end{aligned} \tag{B.2}$$

We choose from the second pair F_j^{m+1} and present here for clarity (A.26)

$$\begin{aligned} F_j^{m+1} = & -\frac{1}{4} \frac{q_j^{m+1}}{q_{j+1}^{m+1}} \frac{\alpha_j^{m+1}}{\alpha_j^m} \left| y_j^{m+1} \right|^2 RP_{j+1}^{m+1} - \frac{1}{4} \frac{q_j^{m+1}}{q_{j+1}^{m+1}} \frac{\alpha_j^{m+1}}{\alpha_j^m} \left| y_j^{m+1} \right|^2 RM_j^{m+1} \\ & + \frac{1}{4} \frac{q_j^{m+1}}{q_{j+1}^{m+1}} \frac{\alpha_j^{m+1}}{\alpha_j^m} \left| y_j^{m+1} \right|^2 \left(\tau_{j+1}^m - \tau_{j+1}^{m+1}, y_{j+1}^{m+1} - y_j^{m+1} \right) \\ & + \frac{1}{2} \frac{q_j^{m+1}}{q_{j+1}^{m+1}} \frac{1}{\alpha_j^m} RM_j^{m+1} \left(\tau_{j+1}^m, y_{j+1}^{m+1} - y_j^{m+1} \right) \\ & + \frac{1}{2} \frac{q_j^{m+1}}{q_{j+1}^{m+1}} \frac{1}{\alpha_j^m} \left(\tau_{j+1}^m - \tau_{j+1}^{m+1}, y_j^{m+1} \right) \left(\tau_{j+1}^m, y_{j+1}^{m+1} - y_j^{m+1} \right) - \frac{1}{4} \frac{\alpha_j^{m+1}}{\alpha_j^m} \left| y_j^{m+1} \right|^2 RP_j^{m+1} \\ & - \frac{1}{4} \frac{\alpha_j^{m+1}}{\alpha_j^m} \left| y_j^{m+1} \right|^2 RM_{j-1}^{m+1} + \frac{1}{4} \frac{\alpha_j^{m+1}}{\alpha_j^m} \left| y_j^{m+1} \right|^2 \left(\tau_j^m - \tau_j^{m+1}, y_j^{m+1} - y_{j-1}^{m+1} \right) \\ & + \frac{1}{2} \frac{1}{\alpha_j^m} RP_j^{m+1} \left(\tau_j^m, y_j^{m+1} - y_{j-1}^{m+1} \right) - \frac{1}{2} \frac{1}{\alpha_j^m} \left(\tau_j^m - \tau_j^{m+1}, y_j^{m+1} \right) \left(\tau_j^m, y_j^{m+1} - y_{j-1}^{m+1} \right) \\ & - \frac{1}{4} \frac{q_j^{m+1}}{\alpha_j^m} RM_j^{m+1} J_{j+1}^{m+1} - \frac{1}{4} \frac{q_j^{m+1}}{\alpha_j^m} RP_j^{m+1} J_j^{m+1} \\ & + \frac{1}{2} \frac{q_j^{m+1}}{q_{j+1}^{m+1}} \frac{1}{\alpha_j^m} \left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^{m+1} - \tau_{j+1}^m \right) \left\{ \frac{1}{2} \alpha_j^{m+1} \left| y_j^{m+1} \right|^2 + RM_j^{m+1} \right\} \\ & + \frac{1}{4} \frac{q_j^{m+1}}{q_{j+1}^{m+1}} \frac{1}{\alpha_j^m} \left(y_{j+1}^{m+1} - y_j^{m+1}, \tau_{j+1}^m \right) \left| \tau_{j+1}^{m+1} - \tau_{j+1}^m \right|^2 \left\{ \frac{1}{2} \alpha_j^{m+1} \left| y_j^{m+1} \right|^2 + RM_j^{m+1} \right\} \\ & - \frac{1}{2} \frac{1}{\alpha_j^m} \left(y_j^{m+1} - y_{j-1}^{m+1}, \tau_j^{m+1} - \tau_j^m \right) \left\{ -\frac{1}{2} \alpha_j^{m+1} \left| y_j^{m+1} \right|^2 - RP_j^{m+1} \right\} \\ & - \frac{1}{4} \frac{1}{\alpha_j^m} \left(y_j^{m+1} - y_{j-1}^{m+1}, \tau_j^m \right) \left| \tau_j^{m+1} - \tau_j^m \right|^2 \left\{ -\frac{1}{2} \alpha_j^{m+1} \left| y_j^{m+1} \right|^2 - RP_j^{m+1} \right\} \\ & + \frac{\lambda}{2} \frac{q_j^{m+1}}{\alpha_j^m} RM_j^{m+1} + \frac{\lambda}{2} \frac{q_j^{m+1}}{\alpha_j^m} RP_j^{m+1} = \sum_{i=1}^{22} Q_i, \end{aligned}$$

where the terms in curly brackets are counted as two independent terms. Hence,

$$\begin{aligned}
|Q_1| &\le C\left|RP_{j+1}^{m+1}\right|,\\
|Q_2|,|Q_{11}|,|Q_{21}| &\le C\left|RM_j^{m+1}\right|,\\
|Q_6|,|Q_{12}|,|Q_{22}| &\le C\left|RP_j^{m+1}\right|,\\
|Q_7| &\le C\left|RM_{j-1}^{m+1}\right|,\\
|Q_3|,|Q_{13}| &\le Ch\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|,\\
|Q_4| &\le C\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|RM_j^{m+1}\right|,\\
|Q_5| &\le C\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|,\\
|Q_8|,|Q_{17}| &\le Ch\left|y_{hu}^{m+1}\right|_{|I_j}\left|\tau_j^{m+1}-\tau_j^{m}\right|,\\
|Q_9| &\le C\left|y_{hu}^{m+1}\right|_{|I_j}\left|RP_j^{m+1}\right|,\\
|Q_{10}| &\le C\left|y_{hu}^{m+1}\right|_{|I_j}\left|\tau_j^{m+1}-\tau_j^{m}\right|,\\
|Q_{14}| &\le C\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|\left|RM_j^{m+1}\right|,\\
|Q_{15}| &\le Ch\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|^2,\\
|Q_{16}| &\le C\left|y_{hu}^{m+1}\right|_{|I_{j+1}}\left|\tau_{j+1}^{m+1}-\tau_{j+1}^{m}\right|^2\left|RM_j^{m+1}\right|,\\
|Q_{18}| &\le C\left|y_{hu}^{m+1}\right|_{|I_j}\left|\tau_j^{m+1}-\tau_j^{m}\right|\left|RP_j^{m+1}\right|,\\
|Q_{19}| &\le Ch\left|y_{hu}^{m+1}\right|_{|I_j}\left|\tau_j^{m+1}-\tau_j^{m}\right|^2,\\
|Q_{20}| &\le C\left|y_{hu}^{m+1}\right|_{|I_j}\left|\tau_j^{m+1}-\tau_j^{m}\right|^2\left|RP_j^{m+1}\right|.
\end{aligned}$$

Using the similar arguments as for D_j^{m+1} we compare the groups to determine the terms of the lower order. We observe that due to Q_5 we can neglect the terms Q_3, Q_{13} and Q_{15} and deal with Q_5 only. Estimating next the difference of the tangents by a constant in Q_{14} and Q_{16} one obtains the term Q_4. Moreover, we can ignore the terms Q_8, Q_{17} and Q_{19} and treat Q_{10}. Analogously, instead of considering Q_{18} and Q_{20} one can examine Q_9. Due to the definition of RP_j^{m+1}, RM_j^{m+1} and notation $I_{(j)}$, which will be used at the end, we choose from the first four groups one representative, for instance Q_6. For the same reason, we estimate Q_9 and Q_{10} instead of Q_4 and Q_5, respectively. Summarizing, we end up with the following terms Q_6, Q_9 and Q_{10}. In view of the above estimates, recalling (4.97) and using (2.12) we arrive at

$$\begin{aligned}
|Q_6|+|Q_9| &\le Ch^{-1}\left(1+\left\|y_{hu}^{m+1}\right\|_{L^\infty}\right)\left|R_j^{m+1}\right|,\\
|Q_{10}| &\le Ch^{-\frac{1}{2}}\left\|y_{hu}^{m+1}\right\|_{L^\infty}\left\|\tau_h^{m+1}-\tau_h^{m}\right\|_{L^2(I_{j+1})}.
\end{aligned}\tag{B.3}$$

We note that due to our remark at the beginning of the proof, E_j^{m+1} and G_j^{m+1} can be estimated in a similar way. Using now the notation $I_{(j)}$, results (B.2), (B.3) and estimate (4.99) we immediately obtain the claim of the lemma.

□

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